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Title: (U) Analytic One-Group S2 Slab Problem with Isotropic Scattering and Fission Applied to Neutron Multiplicity Sensitivity

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SUBJECT: (U) Analytic One-Group S_2 Slab Problem with Isotropic Scattering and Fission Applied to Neutron Multiplicity Sensitivity

I. Introduction

A one-group isotropic S_2 transport problem in a homogeneous slab is solved analytically and used to derive an analytic expression for the Feynman Y (Ref. 1) for this special case. Analytic derivatives of the Feynman Y are then taken with respect to an arbitrary input parameter. The analytic solution is used to verify the results of the new Feynman Y sensitivity capability that Alex Clark (NC State/XCP-3) has integrated into the neutron sensitivity code SENSMG (Refs. 2 and 3). SENSMG's Feynman Y capability relies on the multigroup discrete-ordinates code PARTISN (Ref. 4) and the methodology developed by Mattingly[5] to compute the Feynman Y deterministically,[6] and the Feynman Y sensitivity capability relies on work by O'Brien et al.,[7] extended by Clark.[8,9]

We have previously developed an analytic S_2 transport problem[10] for verification of PARTISN's treatment of upscattering[11] and negative sources.[12] The goal of all of these S_2 solutions is to verify the discrete ordinates coding, including the quadratures and other logic in SENSMG. Thus, an analytic solution of the S_2 problem is used rather than an analytic solution of the continuous-angle transport problem.

The next section of this report derives the Feynman Y for a homogeneous one-group slab with S_2 quadrature and isotropic scattering and fission. Section III derives the first derivative with respect to an arbitrary input parameter that is a material property, and Sec. IV derives the first derivative with respect to the slab width. Section V compares SENSMG results with the analytic results for a two-isotope test problem. Section VI is a summary and conclusions.

II. Derivation

Consider a homogeneous slab of width r_d. The material has a constant isotropic neutron source rate density q. We consider one neutron energy group. Scattering is isotropic. The induced-fission spectrum in one group is 1, but because we will seek the unconstrained sensitivity,[13] we will carry it along in the equations. The quantity of interest is the leakage from the right side of the slab convolved with a response function.

We consider two directions, right and left, with μ_+ the right-going direction cosine and μ_- the left-going. The directions are constrained to satisfy $\mu_+ = -\mu_-$. This problem is then a regular S_2 discrete ordinates calculation.

The equations for the forward right-going and left-going fluxes are

$$\mu_+ \frac{\partial \psi_+(r)}{\partial r} + \Sigma_t \psi_+(r) - \tfrac{1}{2}\Sigma_s(\psi_+(r)+\psi_-(r)) - \tfrac{1}{2}\chi\nu\Sigma_f(\psi_+(r)+\psi_-(r)) = q \tag{1a}$$

and

$$\mu_- \frac{\partial \psi_-(r)}{\partial r} + \Sigma_t \psi_-(r) - \tfrac{1}{2}\Sigma_s(\psi_+(r)+\psi_-(r)) - \tfrac{1}{2}\chi\nu\Sigma_f(\psi_+(r)+\psi_-(r)) = q, \tag{1b}$$

with vacuum boundary conditions

$$\psi_+(-\tfrac{1}{2}r_d) = 0 \tag{2a}$$

and

$$\psi_-(\tfrac{1}{2}r_d) = 0. \tag{2b}$$

(The coordinate system is centered in the middle of the slab.) In Eqs. (1a) and (1b), Σ_t, Σ_s, $\nu\Sigma_f$, and χ are the material total cross section, scattering cross section, product of the number of neutrons per fission and fission cross section, and induced-fission spectrum, respectively.

The equations for the adjoint right-going and left-going fluxes are

$$\mu_+ \frac{\partial \psi_+^*(r)}{\partial r} + \Sigma_t \psi_+^*(r) - \tfrac{1}{2}\Sigma_s(\psi_+^*(r)+\psi_-^*(r)) - \tfrac{1}{2}\chi\nu\Sigma_f(\psi_+^*(r)+\psi_-^*(r)) = 0 \tag{3a}$$

and

$$\mu_- \frac{\partial \psi_-^*(r)}{\partial r} + \Sigma_t \psi_-^*(r) - \tfrac{1}{2}\Sigma_s(\psi_+^*(r)+\psi_-^*(r)) - \tfrac{1}{2}\chi\nu\Sigma_f(\psi_+^*(r)+\psi_-^*(r)) = 0, \tag{3b}$$

with a vacuum boundary condition on the left,

$$\psi_+(-\tfrac{1}{2}r_d) = 0, \tag{4a}$$

and a source on the right,

$$\psi_-(\tfrac{1}{2}r_d) = \Sigma_d. \tag{4b}$$

These adjoint equations [Eqs. (3a) and (3b)] do not have the usual negative sign in front of the spatial derivative term because these are the computational equations (i.e. the equations that will actually be solved) obtained by replacing μ with $-\mu$ and recognizing that adjoint particles travel backwards. Thus, “left-going” and “right-going” here are in the computational sense, not the mathematical sense, in that right-going computational adjoint particles are really going to the right.

First rearrange Eq. (1a) and take the derivative with respect to r to yield

$$\mu_+ \frac{\partial^2 \psi_+(r)}{\partial r^2} + \left(\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f\right)\frac{\partial \psi_+(r)}{\partial r} = \frac{1}{2}\left(\Sigma_s + \chi\nu\Sigma_f\right)\frac{\partial \psi_-(r)}{\partial r}. \tag{5}$$

Rearrange Eq. (1b) to yield

$$\frac{\partial \psi_-(r)}{\partial r} = \frac{1}{\mu_-}q - \frac{1}{\mu_-}\left(\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f\right)\psi_-(r) + \frac{1}{2\mu_-}\left(\Sigma_s + \chi\nu\Sigma_f\right)\psi_+(r), \tag{6}$$

and use Eq. (6) in Eq. (5) to yield

$$\begin{aligned} &\mu_+ \frac{\partial^2 \psi_+(r)}{\partial r^2} + \left(\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f\right)\frac{\partial \psi_+(r)}{\partial r} - \frac{1}{4\mu_-}\left(\Sigma_s + \chi\nu\Sigma_f\right)^2 \psi_+(r) \\ &= \frac{1}{2\mu_-}\left(\Sigma_s + \chi\nu\Sigma_f\right)q - \frac{1}{2\mu_-}\left(\Sigma_s + \chi\nu\Sigma_f\right)\left(\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f\right)\psi_-(r). \end{aligned} \tag{7}$$

Rearrange Eq. (1a) to yield

$$\psi_-(r) = \frac{2}{\left(\Sigma_s + \chi\nu\Sigma_f\right)}\left[-q + \mu_+ \frac{\partial \psi_+(r)}{\partial r} + \left(\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f\right)\psi_+(r)\right], \tag{8}$$

and use Eq. (8) in Eq. (7) (and recognize that $\mu_+ = -\mu_-$) to yield

$$\frac{\partial^2 \psi_+(r)}{\partial r^2} - \frac{\Sigma_t}{\mu_+^2}\left(\Sigma_t - \Sigma_s - \chi\nu\Sigma_f\right)\psi_+(r) = -\frac{\Sigma_t}{\mu_+^2}q. \tag{9}$$

The right-going forward flux is

$$\psi_+(r) = c_1\cos(\lambda r) + c_2\sin(\lambda r) + \psi_P, \tag{10}$$

where the particular solution ψ_P is

$$\psi_P = \frac{q}{\Sigma_t - \Sigma_s - \chi\nu\Sigma_f} \tag{11}$$

and λ is

$$\lambda = \frac{1}{\mu_+}\sqrt{-\Sigma_t(\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)}. \tag{12}$$

The negative sign precedes the first Σ_t because, for the problem of this paper, the term in the parentheses is negative. The trigonometric solution of Eq. (10) accounts for the imaginary roots of the characteristic equation.

The neutron source rate density q is

$$q = \sum_{i=1}^{I} q_i N_i + q_{(\alpha,\mathrm{n})}, \tag{13}$$

where N_i is the atom density of isotope i and q_i is the neutron source rate from the spontaneous fission of isotope i per atom of isotope i (or per 10^{24} atoms, depending on the units of N_i). The simplest way to obtain q_i is to use the neutron source rate density from the spontaneous fission of isotope i (in units of neutrons/cm^3·s) and divide by the atom density of isotope i. Also in Eq. (13), $q_{(\alpha,\mathrm{n})}$ is the source rate density of (α,n) neutrons. The test problem will not include (α,n) neutrons.

Evaluating Eq. (1a) at $r = \frac{1}{2}r_d$ (the right boundary) with Eq. (2b) and using Eq. (2a) leads to the following system of equations for the constants c_1 and c_2:

$$\begin{bmatrix} -\mu_+\lambda\sin(\tfrac{1}{2}\lambda r_d)+(\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d) & \mu_+\lambda\cos(\tfrac{1}{2}\lambda r_d)+(\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\sin(\tfrac{1}{2}\lambda r_d) \\ \cos(\tfrac{1}{2}\lambda r_d) & -\sin(\tfrac{1}{2}\lambda r_d) \end{bmatrix} \times \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -(\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\psi_P+q \\ -\psi_P \end{bmatrix}. \tag{14}$$

The determinant of the coefficient matrix of Eq. (14) is

$$\begin{aligned} D &= \mu_+\lambda\sin^2(\tfrac{1}{2}\lambda r_d)-(\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d)\sin(\tfrac{1}{2}\lambda r_d) \\ &\quad -\mu_+\lambda\cos^2(\tfrac{1}{2}\lambda r_d)-(\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\sin(\tfrac{1}{2}\lambda r_d)\cos(\tfrac{1}{2}\lambda r_d) \\ &= \mu_+\lambda\left(\sin^2(\tfrac{1}{2}\lambda r_d)-\cos^2(\tfrac{1}{2}\lambda r_d)\right)-(\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\sin(\lambda r_d) \\ &= \mu_+\lambda\left(\tfrac{1}{2}-\tfrac{1}{2}\cos(\lambda r_d)-\tfrac{1}{2}-\tfrac{1}{2}\cos(\lambda r_d)\right)-(\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\sin(\lambda r_d) \\ &= -\mu_+\lambda\cos(\lambda r_d)-(\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\sin(\lambda r_d). \end{aligned} \tag{15}$$

Using Cramer's rule, c_1 and c_2 are

$$\begin{aligned} c_1 &= \frac{1}{D}\Big[(\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\psi_P\sin(\tfrac{1}{2}\lambda r_d)-q\sin(\tfrac{1}{2}\lambda r_d) \\ &\quad +\mu_+\lambda\cos(\tfrac{1}{2}\lambda r_d)\psi_P+(\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\sin(\tfrac{1}{2}\lambda r_d)\psi_P\Big] \\ &= \frac{1}{D}\Big[2(\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\psi_P\sin(\tfrac{1}{2}\lambda r_d)-q\sin(\tfrac{1}{2}\lambda r_d)+\mu_+\lambda\cos(\tfrac{1}{2}\lambda r_d)\psi_P\Big] \end{aligned} \tag{16}$$

and

$$\begin{aligned} c_2 &= \frac{1}{D}\Big[\mu_+\lambda\sin(\tfrac{1}{2}\lambda r_d)\psi_P-(\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d)\psi_P \\ &\quad +(\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\psi_P\cos(\tfrac{1}{2}\lambda r_d)-q\cos(\tfrac{1}{2}\lambda r_d)\Big] \\ &= \frac{1}{D}\left(\mu_+\lambda\sin(\tfrac{1}{2}\lambda r_d)\psi_P-q\cos(\tfrac{1}{2}\lambda r_d)\right). \end{aligned} \tag{17}$$

Using $q=(\Sigma_t-\Sigma_s-\chi\nu\Sigma_f)\psi_P$ from Eq. (11), c_1 and c_2 are

$$c_1=\frac{\psi_P}{D}\left(\Sigma_t\sin(\tfrac{1}{2}\lambda r_d)+\mu_+\lambda\cos(\tfrac{1}{2}\lambda r_d)\right) \tag{18}$$

and

$$c_2=\frac{\psi_P}{D}\Big[\mu_+\lambda\sin(\tfrac{1}{2}\lambda r_d)-(\Sigma_t-\Sigma_s-\chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d)\Big]. \tag{19}$$

By symmetry, the left-going forward flux is

$$\psi_-(r)=c_1\cos(\lambda r)-c_2\sin(\lambda r)+\psi_P. \tag{20}$$

The steps leading to Eq. (9) from Eqs. (1a) and (1b) can be done with Eqs. (3a) and (3b) to lead to the following equation for the right-going adjoint flux:

$$\frac{\partial^2\psi_+^*(r)}{\partial r^2}-\frac{\Sigma_t}{\mu_+^2}\left(\Sigma_t-\Sigma_s-\chi\nu\Sigma_f\right)\psi_+^*(r)=0. \tag{21}$$

The right-going adjoint flux is

$$\psi_+^*(r)=c_3\cos(\lambda r)+c_4\sin(\lambda r). \tag{22}$$

Evaluating Eq. (3a) at $r = \frac{1}{2} r_d$ (the right boundary) with Eq. (4b) and using Eq. (4a) leads to the following system of equations for the constants c_3 and c_4:

$$\begin{bmatrix} -\mu_+ \lambda \sin(\tfrac{1}{2}\lambda r_d) + (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d) & \mu_+ \lambda \cos(\tfrac{1}{2}\lambda r_d) + (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\sin(\tfrac{1}{2}\lambda r_d) \\ \cos(\tfrac{1}{2}\lambda r_d) & -\sin(\tfrac{1}{2}\lambda r_d) \end{bmatrix} \times \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2}(\Sigma_s + \chi\nu\Sigma_f)\Sigma_d \\ 0 \end{bmatrix}. \tag{23}$$

The coefficient matrix of Eq. (23) is the same as the coefficient matrix of Eq. (14), so the determinant is the same, D of Eq. (15). Using Cramer's rule, c_3 and c_4 are

$$c_3 = \frac{-\Sigma_d}{2D}(\Sigma_s + \chi\nu\Sigma_f)\sin(\tfrac{1}{2}\lambda r_d) \tag{24}$$

and

$$c_4 = \frac{-\Sigma_d}{2D}(\Sigma_s + \chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d). \tag{25}$$

Similarly, the left-going adjoint flux is

$$\psi_-^*(r) = c_5 \cos(\lambda r) + c_6 \sin(\lambda r), \tag{26}$$

where c_5 and c_6 are the solution of

$$\begin{bmatrix} -\mu_+ \lambda \sin(\tfrac{1}{2}\lambda r_d) + (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d) & -\mu_+ \lambda \cos(\tfrac{1}{2}\lambda r_d) - (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\sin(\tfrac{1}{2}\lambda r_d) \\ \cos(\tfrac{1}{2}\lambda r_d) & \sin(\tfrac{1}{2}\lambda r_d) \end{bmatrix} \times \begin{bmatrix} c_5 \\ c_6 \end{bmatrix} = \begin{bmatrix} 0 \\ \Sigma_d \end{bmatrix}. \tag{27}$$

The determinant of the coefficient matrix of Eq. (27) is

$$\begin{aligned} D_{(26)} &= -\mu_+ \lambda \sin^2(\tfrac{1}{2}\lambda r_d) + (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d)\sin(\tfrac{1}{2}\lambda r_d) \\ &\quad + \mu_+ \lambda \cos^2(\tfrac{1}{2}\lambda r_d) + (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\sin(\tfrac{1}{2}\lambda r_d)\cos(\tfrac{1}{2}\lambda r_d) \\ &= -\mu_+ \lambda \left(\sin^2(\tfrac{1}{2}\lambda r_d) - \cos^2(\tfrac{1}{2}\lambda r_d)\right) + (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\sin(\lambda r_d) \\ &= -D. \end{aligned} \tag{28}$$

Using Cramer's rule, c_5 and c_6 are

$$\begin{aligned} c_5 &= \frac{1}{D_{(26)}}\left[\mu_+ \lambda \cos(\tfrac{1}{2}\lambda r_d)\Sigma_d + (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\sin(\tfrac{1}{2}\lambda r_d)\Sigma_d\right] \\ &= \frac{-\Sigma_d}{D}\left[\mu_+ \lambda \cos(\tfrac{1}{2}\lambda r_d) + (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\sin(\tfrac{1}{2}\lambda r_d)\right] \end{aligned} \tag{29}$$

and

$$\begin{aligned} c_6 &= \frac{1}{D_{(26)}}\left[-\mu_+ \lambda \sin(\tfrac{1}{2}\lambda r_d)\Sigma_d + (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d)\Sigma_d\right] \\ &= \frac{-\Sigma_d}{D}\left[-\mu_+ \lambda \sin(\tfrac{1}{2}\lambda r_d) + (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d)\right]. \end{aligned} \tag{30}$$

The first moment R_1 of the count rate distribution, which is the usual singles count rate, is

$$R_1 = \tfrac{1}{2}\Sigma_d \mu_+ \psi_+(\tfrac{1}{2} r_d). \tag{31}$$

The second moment R_2 of the count rate distribution is

$$R_2 = {}_2S + {}_2S_{s.f.}, \tag{32}$$

where

$${}_2S = \int_{-r_d/2}^{r_d/2} dr \overline{\nu(\nu-1)}\Sigma_f \left(\chi\phi^*(r)\right)^2 \phi(r) = \overline{\nu(\nu-1)}\Sigma_f \chi^2 \int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r)\right)^2 \phi(r) \tag{33}$$

and

$${}_2S_{s.f.} = \int_{-r_d/2}^{r_d/2} dr \left(\frac{\overline{\nu(\nu-1)}}{\overline{\nu}}\right)_{s.f.} q\left(\chi_{s.f.}\phi^*(r)\right)^2 = \left(\frac{\overline{\nu(\nu-1)}}{\overline{\nu}}\right)_{s.f.} q\chi_{s.f.}^2 \int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r)\right)^2. \tag{34}$$

In Eqs. (33) and (34), χ is the induced-fission spectrum as before and $\chi_{s.f.}$ is the spontaneous-fission spectrum. Using PARTISN, a vector χ, and the Nuclear Data Interface (NDI) at LANL, the induced-fission spectrum is defined for mixtures (in a one-group problem) as

$$\chi = \frac{\sum_{i=1}^{I} \chi_i \nu\sigma_{f,i} N_i f_i}{\sum_{i=1}^{I} \nu\sigma_{f,i} N_i f_i}, \tag{35}$$

where f_i is the spectrum weighting function and I is the number of fissionable isotopes. If the NDI is not used or if a matrix χ is used, $f_i = 1$. For the one-group problem, $\chi_{s.f.} = 1$.

Also in Eqs. (33) and (34), $\overline{\nu}$ and $\overline{\nu(\nu-1)}$ are the first and second factorial moments of the fission multiplicity distributions. These are given as isotopic nuclear data.[14,15] The products $\overline{\nu(\nu-1)}\Sigma_f$ and $\left(\overline{\nu(\nu-1)}/\overline{\nu}\right)_{s.f.} q$ are defined for mixtures as

$$\overline{\nu(\nu-1)}\Sigma_f = \sum_{i=1}^{I} N_i \overline{\nu(\nu-1)}_i \sigma_{f,i} \tag{36}$$

and

$$\left(\frac{\overline{\nu(\nu-1)}}{\overline{\nu}}\right)_{s.f.} q = \sum_{i=1}^{I} N_i \left(\frac{\overline{\nu(\nu-1)}}{\overline{\nu}}\right)_{s.f.,i} q_i. \tag{37}$$

Only isotopes with data for the moments of the multiplicity distributions will contribute to the material quantities $\overline{\nu(\nu-1)}\Sigma_f$ and $\left(\overline{\nu(\nu-1)}/\overline{\nu}\right)_{s.f.} q$.

The Feynman Y asymptote is

$$Y = \frac{R_2}{R_1}. \tag{38}$$

The forward and adjoint scalar fluxes are

$$\phi(r) = \tfrac{1}{2}\left(\psi_+(r) + \psi_-(r)\right) = c_1 \cos(\lambda r) + \psi_P \tag{39}$$

and

$$\phi^*(r) = \tfrac{1}{2}\left(\psi_+^*(r) + \psi_-^*(r)\right) = \tfrac{1}{2}\left((c_3 + c_5)\cos(\lambda r) + (c_4 + c_6)\sin(\lambda r)\right), \tag{40}$$

respectively.

The volume integral of the square of the adjoint scalar flux is

$$\begin{aligned}\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2 &= \frac{1}{4}\int_{-r_d/2}^{r_d/2} dr\left((c_3+c_5)\cos(\lambda r)+(c_4+c_6)\sin(\lambda r)\right)^2 \\ &= \frac{1}{4}\int_{-r_d/2}^{r_d/2} dr\Big((c_3+c_5)^2\cos^2(\lambda r)+(c_4+c_6)^2\sin^2(\lambda r) \\ &\quad +2(c_3+c_5)(c_4+c_6)\sin(\lambda r)\cos(\lambda r)\Big) \\ &= \frac{1}{8\lambda}\Big[(c_3+c_5)^2\left(\lambda r_d+\sin(\lambda r_d)\right)+(c_4+c_6)^2\left(\lambda r_d-\sin(\lambda r_d)\right)\Big].\end{aligned} \tag{41}$$

The volume integral of the square of the adjoint scalar flux multiplied by the forward scalar flux is

$$\begin{aligned}\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2\phi(r) &= \frac{1}{4}\int_{-r_d/2}^{r_d/2} dr\Big((c_3+c_5)^2\cos^2(\lambda r)+(c_4+c_6)^2\sin^2(\lambda r) \\ &\quad +2(c_3+c_5)(c_4+c_6)\sin(\lambda r)\cos(\lambda r)\Big)\left(c_1\cos(\lambda r)+\psi_P\right) \\ &= \frac{c_1}{4}\int_{-r_d/2}^{r_d/2} dr\Big((c_3+c_5)^2\cos^3(\lambda r)+(c_4+c_6)^2\sin^2(\lambda r)\cos(\lambda r) \\ &\quad +2(c_3+c_5)(c_4+c_6)\sin(\lambda r)\cos^2(\lambda r)\Big) \\ &\quad +\psi_P\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2 \\ &= \frac{c_1}{6\lambda}\Big[\tfrac{1}{4}(c_3+c_5)^2\left(9\sin(\tfrac{1}{2}\lambda r_d)+\sin(\tfrac{3}{2}\lambda r_d)\right)+(c_4+c_6)^2\sin^3(\tfrac{1}{2}\lambda r_d)\Big] \\ &\quad +\psi_P\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2.\end{aligned} \tag{42}$$

The inner product of the forward and adjoint angular fluxes is

$$\begin{aligned}\int_{-1}^{1} d\mu\int_{-r_d/2}^{r_d/2} dr\,\psi^*(r,-\mu)\psi(r,\mu) &= \int_{-r_d/2}^{r_d/2} dr\,\tfrac{1}{2}\left(\psi_+^*(r)\psi_-(r)+\psi_-^*(r)\psi_+(r)\right) \\ &= \frac{1}{2}\int_{-r_d/2}^{r_d/2} dr\Big[\left(c_3\cos(\lambda r)+c_4\sin(\lambda r)\right)\left(c_1\cos(\lambda r)-c_2\sin(\lambda r)+\psi_P\right) \\ &\quad +\left(c_5\cos(\lambda r)+c_6\sin(\lambda r)\right)\left(c_1\cos(\lambda r)+c_2\sin(\lambda r)+\psi_P\right)\Big] \\ &= \frac{1}{2}\int_{-r_d/2}^{r_d/2} dr\Big[c_1(\mathrm{c}_3+c_5)\cos^2(\lambda r)+c_2(c_6-c_4)\sin^2(\lambda r) \\ &\quad +\left(c_1(c_4+c_6)+c_2(c_5-c_3)\right)\sin(\lambda r)\cos(\lambda r) \\ &\quad +\psi_P(c_3+c_5)\cos(\lambda r)+\psi_P(c_4+c_6)\sin(\lambda r)\Big] \\ &= \frac{1}{\lambda}\Big[\tfrac{1}{4}c_1(c_3+c_5)\left(\lambda r_d+\sin(\lambda r_d)\right)+\tfrac{1}{4}c_2(c_6-c_4)\left(\lambda r_d-\sin(\lambda r_d)\right) \\ &\quad +\psi_P(c_3+c_5)\sin(\tfrac{1}{2}\lambda r_d)\Big].\end{aligned} \tag{43}$$

III. Derivatives with Respect to an Arbitrary Input Parameter (Material Property)

We seek the derivative of the Feynman Y of Eq. (38) with respect to an arbitrary input parameter α_x that is a material property. The derivative with respect to the slab width is worked out in Sec. IV.

To begin, we note that we will need the derivatives of the input parameters appearing in Eqs. (1a) through (4b) and (33) and (34): macroscopic material cross sections Σ_t, Σ_s, and $\nu\Sigma_f$; the material fission spectrum χ; the source rate density q; the slab width r_d; induced-fission neutron number distribution moments $\bar{\nu}$ and $\overline{\nu(\nu-1)}$; and spontaneous-fission neutron number distribution moments $\bar{\nu}_{s.f.}$ and $\overline{\nu(\nu-1)}_{s.f.}$. In this memo, we will not take derivatives with respect to the detector response function Σ_d or the spontaneous-fission spectrum $\chi_{s.f.}$. We will also assume the direction cosine μ_+ is known perfectly.

The derivative of ψ_P of Eq. (11) is

$$\begin{aligned}\frac{\partial \psi_P}{\partial \alpha_x} &= \frac{1}{\Sigma_t-\Sigma_s-\chi\nu\Sigma_f}\frac{\partial q}{\partial \alpha_x} - \frac{q}{(\Sigma_t-\Sigma_s-\chi\nu\Sigma_f)^2}\frac{\partial(\Sigma_t-\Sigma_s-\chi\nu\Sigma_f)}{\partial \alpha_x} \\ &= \frac{\psi_P}{q}\frac{\partial q}{\partial \alpha_x} - \frac{\psi_P}{\Sigma_t-\Sigma_s-\chi\nu\Sigma_f}\frac{\partial(\Sigma_t-\Sigma_s-\chi\nu\Sigma_f)}{\partial \alpha_x}.\end{aligned} \tag{44}$$

The derivative of λ of Eq. (12) is

$$\begin{aligned}\frac{\partial \lambda}{\partial \alpha_x} &= \frac{1}{2\mu_+\sqrt{-\Sigma_t(\Sigma_t-\Sigma_s-\chi\nu\Sigma_f)}}\left(-\frac{\partial \Sigma_t}{\partial \alpha_x}(\Sigma_t-\Sigma_s-\chi\nu\Sigma_f) - \Sigma_t\frac{\partial(\Sigma_t-\Sigma_s-\chi\nu\Sigma_f)}{\partial \alpha_x}\right) \\ &= -\frac{\sqrt{-\Sigma_t(\Sigma_t-\Sigma_s-\chi\nu\Sigma_f)}}{2\mu_+\left(-\Sigma_t(\Sigma_t-\Sigma_s-\chi\nu\Sigma_f)\right)}\left(\frac{\partial \Sigma_t}{\partial \alpha_x}(\Sigma_t-\Sigma_s-\chi\nu\Sigma_f) + \Sigma_t\frac{\partial(\Sigma_t-\Sigma_s-\chi\nu\Sigma_f)}{\partial \alpha_x}\right) \\ &= -\frac{\lambda}{2\left(-\Sigma_t(\Sigma_t-\Sigma_s-\chi\nu\Sigma_f)\right)}\left(\frac{\partial \Sigma_t}{\partial \alpha_x}(\Sigma_t-\Sigma_s-\chi\nu\Sigma_f) + \Sigma_t\frac{\partial(\Sigma_t-\Sigma_s-\chi\nu\Sigma_f)}{\partial \alpha_x}\right) \\ &= \frac{\lambda}{2}\left(\frac{1}{\Sigma_t}\frac{\partial \Sigma_t}{\partial \alpha_x} + \frac{1}{(\Sigma_t-\Sigma_s-\chi\nu\Sigma_f)}\frac{\partial(\Sigma_t-\Sigma_s-\chi\nu\Sigma_f)}{\partial \alpha_x}\right).\end{aligned} \tag{45}$$

The derivative of $1/\lambda$ is

$$\frac{\partial}{\partial \alpha_x}\left(\frac{1}{\lambda}\right) = -\frac{1}{\lambda^2}\frac{\partial \lambda}{\partial \alpha_x}. \tag{46}$$

We will need the derivatives of sines and cosines. They are

$$\frac{\partial \sin(a\lambda r_d)}{\partial \alpha_x} = \frac{\partial \sin(a\lambda r_d)}{\partial \lambda}\frac{\partial \lambda}{\partial \alpha_x} = a\, r_d \cos(a\lambda r_d)\frac{\partial \lambda}{\partial \alpha_x} \tag{47}$$

and

$$\frac{\partial \cos(a\lambda r_d)}{\partial \alpha_x} = \frac{\partial \cos(a\lambda r_d)}{\partial \lambda}\frac{\partial \lambda}{\partial \alpha_x} = -a\, r_d \sin(a\lambda r_d)\frac{\partial \lambda}{\partial \alpha_x}, \tag{48}$$

where a is an arbitrary coefficient that does not depend on α_x.

The derivative of D of Eq. (15) is

$$\begin{aligned}
\frac{\partial D}{\partial \alpha_x} &= -\mu_+ \frac{\partial \lambda}{\partial \alpha_x}\cos(\lambda r_d) - \mu_+ \lambda \frac{\partial \cos(\lambda r_d)}{\partial \alpha_x} \\
&\quad - \frac{\partial(\Sigma_t - \frac{1}{2}\Sigma_s - \frac{1}{2}\chi\nu\Sigma_f)}{\partial \alpha_x}\sin(\lambda r_d) - (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\frac{\partial \sin(\lambda r_d)}{\partial \alpha_x} \\
&= -\mu_+ \frac{\partial \lambda}{\partial \alpha_x}\cos(\lambda r_d) + \mu_+ \lambda r_d \sin(\lambda r_d)\frac{\partial \lambda}{\partial \alpha_x} \\
&\quad - \frac{\partial(\Sigma_t - \frac{1}{2}\Sigma_s - \frac{1}{2}\chi\nu\Sigma_f)}{\partial \alpha_x}\sin(\lambda r_d) - (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f) r_d \cos(\lambda r_d)\frac{\partial \lambda}{\partial \alpha_x} \\
&= \left[-\mu_+ \cos(\lambda r_d) + \mu_+ \lambda r_d \sin(\lambda r_d) - (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f) r_d \cos(\lambda r_d)\right]\frac{\partial \lambda}{\partial \alpha_x} \\
&\quad - \frac{\partial(\Sigma_t - \frac{1}{2}\Sigma_s - \frac{1}{2}\chi\nu\Sigma_f)}{\partial \alpha_x}\sin(\lambda r_d) \\
&= \left[-\left(\mu_+ + (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f) r_d\right)\cos(\lambda r_d) + \mu_+ \lambda r_d \sin(\lambda r_d)\right]\frac{\partial \lambda}{\partial \alpha_x} \\
&\quad - \frac{\partial(\Sigma_t - \frac{1}{2}\Sigma_s - \frac{1}{2}\chi\nu\Sigma_f)}{\partial \alpha_x}\sin(\lambda r_d).
\end{aligned} \tag{49}$$

The derivative of $1/D$ is

$$\frac{\partial}{\partial \alpha_x}\left(\frac{1}{D}\right) = -\frac{1}{D^2}\frac{\partial D}{\partial \alpha_x}. \tag{50}$$

The derivative of c_1 of Eq. (18) is

$$
\begin{aligned}
\frac{\partial c_1}{\partial \alpha_x} &= \frac{\partial \psi_P}{\partial \alpha_x}\frac{1}{D}\left(\Sigma_t \sin(\tfrac{1}{2}\lambda r_d) + \mu_+\lambda\cos(\tfrac{1}{2}\lambda r_d)\right) \\
&\quad + \psi_P \frac{\partial}{\partial \alpha_x}\left(\frac{1}{D}\right)\left(\Sigma_t \sin(\tfrac{1}{2}\lambda r_d) + \mu_+\lambda\cos(\tfrac{1}{2}\lambda r_d)\right) \\
&\quad + \frac{\psi_P}{D}\left(\frac{\partial \Sigma_t}{\partial \alpha_x}\sin(\tfrac{1}{2}\lambda r_d) + \Sigma_t\frac{\partial \sin(\tfrac{1}{2}\lambda r_d)}{\partial \alpha_x} + \mu_+\frac{\partial \lambda}{\partial \alpha_x}\cos(\tfrac{1}{2}\lambda r_d) + \mu_+\lambda\frac{\partial \cos(\tfrac{1}{2}\lambda r_d)}{\partial \alpha_x}\right) \\
&= \frac{\partial \psi_P}{\partial \alpha_x}\frac{1}{D}\left(\Sigma_t \sin(\tfrac{1}{2}\lambda r_d) + \mu_+\lambda\cos(\tfrac{1}{2}\lambda r_d)\right) \\
&\quad - \frac{\psi_P}{D^2}\frac{\partial D}{\partial \alpha_x}\left(\Sigma_t \sin(\tfrac{1}{2}\lambda r_d) + \mu_+\lambda\cos(\tfrac{1}{2}\lambda r_d)\right) \\
&\quad + \frac{\psi_P}{D}\left(\frac{\partial \Sigma_t}{\partial \alpha_x}\sin(\tfrac{1}{2}\lambda r_d) + \Sigma_t\tfrac{1}{2}r_d\cos(\tfrac{1}{2}\lambda r_d)\frac{\partial \lambda}{\partial \alpha_x} + \mu_+\frac{\partial \lambda}{\partial \alpha_x}\cos(\tfrac{1}{2}\lambda r_d) - \mu_+\lambda\tfrac{1}{2}r_d\sin(\tfrac{1}{2}\lambda r_d)\frac{\partial \lambda}{\partial \alpha_x}\right) \\
&= \left(\frac{\partial \psi_P}{\partial \alpha_x}\frac{1}{D} - \frac{\psi_P}{D^2}\frac{\partial D}{\partial \alpha_x}\right)\left(\Sigma_t \sin(\tfrac{1}{2}\lambda r_d) + \mu_+\lambda\cos(\tfrac{1}{2}\lambda r_d)\right) \\
&\quad + \frac{\psi_P}{D}\left(\frac{\partial \Sigma_t}{\partial \alpha_x}\sin(\tfrac{1}{2}\lambda r_d) + \Sigma_t\tfrac{1}{2}r_d\cos(\tfrac{1}{2}\lambda r_d)\frac{\partial \lambda}{\partial \alpha_x} + \mu_+\frac{\partial \lambda}{\partial \alpha_x}\cos(\tfrac{1}{2}\lambda r_d) - \mu_+\lambda\tfrac{1}{2}r_d\sin(\tfrac{1}{2}\lambda r_d)\frac{\partial \lambda}{\partial \alpha_x}\right) \\
&= \left(\frac{1}{\psi_P}\frac{\partial \psi_P}{\partial \alpha_x} - \frac{1}{D}\frac{\partial D}{\partial \alpha_x}\right)c_1 + \frac{\psi_P}{D_{(12)}}\frac{\partial \Sigma_t}{\partial \alpha_x}\sin(\tfrac{1}{2}\lambda r_d) \\
&\quad + \frac{\psi_P}{D}\left[\left(\frac{\Sigma_t r_d}{2} + \mu_+\right)\cos(\tfrac{1}{2}\lambda r_d) - \frac{\mu_+\lambda r_d}{2}\sin(\tfrac{1}{2}\lambda r_d)\right]\frac{\partial \lambda}{\partial \alpha_x}. \qquad (51)
\end{aligned}
$$

The derivative of c_2 of Eq. (19) is

$$
\begin{aligned}
\frac{\partial c_2}{\partial \alpha_x} &= \frac{\partial \psi_P}{\partial \alpha_x}\frac{1}{D}\left[\mu_+\lambda\sin(\tfrac{1}{2}\lambda r_d) - (\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d)\right] \\
&\quad + \psi_P \frac{\partial}{\partial \alpha_x}\left(\frac{1}{D}\right)\left[\mu_+\lambda\sin(\tfrac{1}{2}\lambda r_d) - (\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d)\right] \\
&\quad + \frac{\psi_P}{D}\left(\mu_+\frac{\partial\lambda}{\partial\alpha_x}\sin(\tfrac{1}{2}\lambda r_d) + \mu_+\lambda\frac{\partial \sin(\tfrac{1}{2}\lambda r_d)}{\partial \alpha_x}\right. \\
&\quad \left. - \frac{\partial(\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)}{\partial\alpha_x}\cos(\tfrac{1}{2}\lambda r_d) - (\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)\frac{\partial \cos(\tfrac{1}{2}\lambda r_d)}{\partial\alpha_x}\right) \\
&= \frac{\partial \psi_P}{\partial \alpha_x}\frac{1}{D}\left[\mu_+\lambda\sin(\tfrac{1}{2}\lambda r_d) - (\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d)\right] \\
&\quad - \frac{\psi_P}{D^2}\frac{\partial D}{\partial \alpha_x}\left[\mu_+\lambda\sin(\tfrac{1}{2}\lambda r_d) - (\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d)\right] \\
&\quad + \frac{\psi_P}{D}\left(\mu_+\frac{\partial\lambda}{\partial\alpha_x}\sin(\tfrac{1}{2}\lambda r_d) + \mu_+\lambda\tfrac{1}{2}r_d\cos(\tfrac{1}{2}\lambda r_d)\frac{\partial\lambda}{\partial\alpha_x}\right. \\
&\quad \left. - \frac{\partial(\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)}{\partial\alpha_x}\cos(\tfrac{1}{2}\lambda r_d) + (\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)\tfrac{1}{2}r_d\sin(\tfrac{1}{2}\lambda r_d)\frac{\partial\lambda}{\partial\alpha_x}\right) \\
&= \left(\frac{\partial \psi_P}{\partial \alpha_x}\frac{1}{D} - \frac{\psi_P}{D^2}\frac{\partial D}{\partial \alpha_x}\right)\left[\mu_+\lambda\sin(\tfrac{1}{2}\lambda r_d) - (\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d)\right] \\
&\quad + \frac{\psi_P}{D}\left(\mu_+\frac{\partial\lambda}{\partial\alpha_x}\sin(\tfrac{1}{2}\lambda r_d) + \mu_+\lambda\tfrac{1}{2}r_d\cos(\tfrac{1}{2}\lambda r_d)\frac{\partial\lambda}{\partial\alpha_x}\right. \\
&\quad \left. - \frac{\partial(\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)}{\partial\alpha_x}\cos(\tfrac{1}{2}\lambda r_d) + (\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)\tfrac{1}{2}r_d\sin(\tfrac{1}{2}\lambda r_d)\frac{\partial\lambda}{\partial\alpha_x}\right) \\
&= \left(\frac{1}{\psi_P}\frac{\partial \psi_P}{\partial \alpha_x} - \frac{1}{D}\frac{\partial D}{\partial \alpha_x}\right)c_2 - \frac{\psi_P}{D}\frac{\partial(\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)}{\partial\alpha_x}\cos(\tfrac{1}{2}\lambda r_d) \\
&\quad + \frac{\psi_P}{D}\left[\left(\mu_+ + \frac{r_d}{2}(\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)\right)\sin(\tfrac{1}{2}\lambda r_d) + \frac{\mu_+\lambda r_d}{2}\cos(\tfrac{1}{2}\lambda r_d)\right]\frac{\partial\lambda}{\partial\alpha_x}.
\end{aligned}
\tag{52}
$$

The derivative of c_3 of Eq. (24) is

$$\begin{aligned}
\frac{\partial c_3}{\partial \alpha_x} &= \frac{-\Sigma_d}{2} \frac{\partial}{\partial \alpha_x}\left(\frac{1}{D}\right)(\Sigma_s + \chi\nu\Sigma_f)\sin(\tfrac{1}{2}\lambda r_d) - \frac{\Sigma_d}{2D}\frac{\partial(\Sigma_s + \chi\nu\Sigma_f)}{\partial \alpha_x}\sin(\tfrac{1}{2}\lambda r_d) \\
&\quad - \frac{\Sigma_d}{2D}(\Sigma_s + \chi\nu\Sigma_f)\frac{\partial \sin(\tfrac{1}{2}\lambda r_d)}{\partial \alpha_x} \\
&= \frac{\Sigma_d}{2D^2}\frac{\partial D}{\partial \alpha_x}(\Sigma_s + \chi\nu\Sigma_f)\sin(\tfrac{1}{2}\lambda r_d) - \frac{\Sigma_d}{2D}\sin(\tfrac{1}{2}\lambda r_d)\frac{\partial(\Sigma_s + \chi\nu\Sigma_f)}{\partial \alpha_x} \\
&\quad - \frac{\Sigma_d}{2D}(\Sigma_s + \chi\nu\Sigma_f)\tfrac{1}{2} r_d \cos(\tfrac{1}{2}\lambda r_d)\frac{\partial \lambda}{\partial \alpha_x} \\
&= -\frac{c_3}{D}\frac{\partial D}{\partial \alpha_x} - \frac{\Sigma_d}{2D}\sin(\tfrac{1}{2}\lambda r_d)\frac{\partial(\Sigma_s + \chi\nu\Sigma_f)}{\partial \alpha_x} - \frac{\Sigma_d r_d}{4D}(\Sigma_s + \chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d)\frac{\partial \lambda}{\partial \alpha_x}.
\end{aligned} \tag{53}$$

The derivative of c_4 of Eq. (25) is

$$\begin{aligned}
\frac{\partial c_4}{\partial \alpha_x} &= \frac{-\Sigma_d}{2} \frac{\partial}{\partial \alpha_x}\left(\frac{1}{D}\right)(\Sigma_s + \chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d) - \frac{\Sigma_d}{2D}\frac{\partial(\Sigma_s + \chi\nu\Sigma_f)}{\partial \alpha_x}\cos(\tfrac{1}{2}\lambda r_d) \\
&\quad - \frac{\Sigma_d}{2D}(\Sigma_s + \chi\nu\Sigma_f)\frac{\partial \cos(\tfrac{1}{2}\lambda r_d)}{\partial \alpha_x} \\
&= \frac{\Sigma_d}{2D^2}\frac{\partial D}{\partial \alpha_x}(\Sigma_s + \chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d) - \frac{\Sigma_d}{2D}\cos(\tfrac{1}{2}\lambda r_d)\frac{\partial(\Sigma_s + \chi\nu\Sigma_f)}{\partial \alpha_x} \\
&\quad + \frac{\Sigma_d}{2D}(\Sigma_s + \chi\nu\Sigma_f)\tfrac{1}{2} r_d \sin(\tfrac{1}{2}\lambda r_d)\frac{\partial \lambda}{\partial \alpha_x} \\
&= -\frac{c_4}{D}\frac{\partial D}{\partial \alpha_x} - \frac{\Sigma_d}{2D}\cos(\tfrac{1}{2}\lambda r_d)\frac{\partial(\Sigma_s + \chi\nu\Sigma_f)}{\partial \alpha_x} + \frac{\Sigma_d r_d}{4D}(\Sigma_s + \chi\nu\Sigma_f)\sin(\tfrac{1}{2}\lambda r_d)\frac{\partial \lambda}{\partial \alpha_x}.
\end{aligned} \tag{54}$$

The derivative of c_5 of Eq. (29) is

$$
\begin{aligned}
\frac{\partial c_5}{\partial \alpha_x} &= -\Sigma_d \frac{\partial}{\partial \alpha_x}\left(\frac{1}{D}\right)\left[\mu_+ \lambda \cos(\tfrac{1}{2}\lambda r_d) + (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\sin(\tfrac{1}{2}\lambda r_d)\right] \\
&\quad - \frac{\Sigma_d}{D}\left[\mu_+ \frac{\partial \lambda}{\partial \alpha_x}\cos(\tfrac{1}{2}\lambda r_d) + \mu_+ \lambda \frac{\partial \cos(\tfrac{1}{2}\lambda r_d)}{\partial \alpha_x}\right. \\
&\quad \left. + \frac{\partial(\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)}{\partial \alpha_x}\sin(\tfrac{1}{2}\lambda r_d) + (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\frac{\partial \sin(\tfrac{1}{2}\lambda r_d)}{\partial \alpha_x}\right] \\
&= \frac{\Sigma_d}{D^2}\frac{\partial D}{\partial \alpha_x}\left[\mu_+ \lambda \cos(\tfrac{1}{2}\lambda r_d) + (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\sin(\tfrac{1}{2}\lambda r_d)\right] \\
&\quad - \frac{\Sigma_d}{D}\left[\mu_+ \frac{\partial \lambda}{\partial \alpha_x}\cos(\tfrac{1}{2}\lambda r_d) - \mu_+ \lambda \tfrac{1}{2} r_d \sin(\tfrac{1}{2}\lambda r_d)\frac{\partial \lambda}{\partial \alpha_x}\right. \\
&\quad \left. + \frac{\partial(\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)}{\partial \alpha_x}\sin(\tfrac{1}{2}\lambda r_d) + (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\tfrac{1}{2} r_d \cos(\tfrac{1}{2}\lambda r_d)\frac{\partial \lambda}{\partial \alpha_x}\right] \\
&= -\frac{c_5}{D}\frac{\partial D}{\partial \alpha_x} - \frac{\Sigma_d}{D}\frac{\partial(\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)}{\partial \alpha_x}\sin(\tfrac{1}{2}\lambda r_d) \\
&\quad - \frac{\Sigma_d}{D}\left[\left(\mu_+ + \frac{r_d}{2}(\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\right)\cos(\tfrac{1}{2}\lambda r_d) - \frac{\mu_+ \lambda r_d}{2}\sin(\tfrac{1}{2}\lambda r_d)\right]\frac{\partial \lambda}{\partial \alpha_x}.
\end{aligned}
\tag{55}
$$

The derivative of c_6 of Eq. (30) is

$$
\begin{aligned}
\frac{\partial c_6}{\partial \alpha_x} &= -\Sigma_d \frac{\partial}{\partial \alpha_x}\left(\frac{1}{D}\right)\left[-\mu_+\lambda \sin(\tfrac{1}{2}\lambda r_d) + (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d)\right] \\
&\quad - \frac{\Sigma_d}{D}\left[-\mu_+ \frac{\partial \lambda}{\partial \alpha_x}\sin(\tfrac{1}{2}\lambda r_d) - \mu_+\lambda \frac{\partial \sin(\tfrac{1}{2}\lambda r_d)}{\partial \alpha_x}\right. \\
&\quad \left. + \frac{\partial(\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)}{\partial \alpha_x}\cos(\tfrac{1}{2}\lambda r_d) + (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\frac{\partial \cos(\tfrac{1}{2}\lambda r_d)}{\partial \alpha_x}\right] \\
&= \frac{\Sigma_d}{D^2}\frac{\partial D}{\partial \alpha_x}\left[-\mu_+\lambda \sin(\tfrac{1}{2}\lambda r_d) + (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d)\right] \\
&\quad - \frac{\Sigma_d}{D}\left[-\mu_+ \frac{\partial \lambda}{\partial \alpha_x}\sin(\tfrac{1}{2}\lambda r_d) - \mu_+\lambda \tfrac{1}{2} r_d \cos(\tfrac{1}{2}\lambda r_d)\frac{\partial \lambda}{\partial \alpha_x}\right. \\
&\quad \left. + \frac{\partial(\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)}{\partial \alpha_x}\cos(\tfrac{1}{2}\lambda r_d) - (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\tfrac{1}{2} r_d \sin(\tfrac{1}{2}\lambda r_d)\frac{\partial \lambda}{\partial \alpha_x}\right] \\
&= -\frac{c_6}{D}\frac{\partial D}{\partial \alpha_x} - \frac{\Sigma_d}{D}\frac{\partial(\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)}{\partial \alpha_x}\cos(\tfrac{1}{2}\lambda r_d) \\
&\quad - \frac{\Sigma_d}{D}\left[-\left(\mu_+ + \frac{r_d}{2}(\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\right)\sin(\tfrac{1}{2}\lambda r_d) - \frac{\mu_+\lambda r_d}{2}\cos(\tfrac{1}{2}\lambda r_d)\right]\frac{\partial \lambda}{\partial \alpha_x}.
\end{aligned} \tag{56}
$$

The derivative of $\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2$ of Eq. (41) is

$$
\begin{aligned}
\frac{\partial}{\partial \alpha_x}\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2 &= -\frac{1}{8\lambda^2}\frac{\partial \lambda}{\partial \alpha_x}\left[(c_3+c_5)^2\left(\lambda r_d + \sin(\lambda r_d)\right) + (c_4+c_6)^2\left(\lambda r_d - \sin(\lambda r_d)\right)\right] \\
&\quad + \frac{1}{8\lambda}\left[2(c_3+c_5)\left(\frac{\partial c_3}{\partial \alpha_x} + \frac{\partial c_5}{\partial \alpha_x}\right)\left(\lambda r_d + \sin(\lambda r_d)\right) + (c_3+c_5)^2\left(\frac{\partial \lambda}{\partial \alpha_x}r_d + \frac{\partial \sin(\lambda r_d)}{\partial \alpha_x}\right)\right. \\
&\quad \left. + 2(c_4+c_6)\left(\frac{\partial c_4}{\partial \alpha_x} + \frac{\partial c_6}{\partial \alpha_x}\right)\left(\lambda r_d - \sin(\lambda r_d)\right) + (c_4+c_6)^2\left(\frac{\partial \lambda}{\partial \alpha_x}r_d - \frac{\partial \sin(\lambda r_d)}{\partial \alpha_x}\right)\right] \\
&= -\frac{1}{\lambda}\frac{\partial \lambda}{\partial \alpha_x}\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2 + \frac{1}{4\lambda}\left[(c_3+c_5)\left(\frac{\partial c_3}{\partial \alpha_x} + \frac{\partial c_5}{\partial \alpha_x}\right)\left(\lambda r_d + \sin(\lambda r_d)\right)\right. \\
&\quad \left. + (c_4+c_6)\left(\frac{\partial c_4}{\partial \alpha_x} + \frac{\partial c_6}{\partial \alpha_x}\right)\left(\lambda r_d - \sin(\lambda r_d)\right)\right] \\
&\quad + \frac{r_d}{8\lambda}\left[\left((c_3+c_5)^2 - (c_4+c_6)^2\right)\cos(\lambda r_d) + (c_3+c_5)^2 + (c_4+c_6)^2\right]\frac{\partial \lambda}{\partial \alpha_x}.
\end{aligned} \tag{57}
$$

The derivative of $\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2 \phi(r)$ of Eq. (42) is

$$
\begin{aligned}
\frac{\partial}{\partial\alpha_x}\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2 \phi(r) = {} & \frac{\partial c_1}{\partial\alpha_x}\frac{1}{6\lambda}\left[\tfrac{1}{4}(c_3+c_5)^2\left(9\sin(\tfrac{1}{2}\lambda r_d)+\sin(\tfrac{3}{2}\lambda r_d)\right)+(c_4+c_6)^2\sin^3(\tfrac{1}{2}\lambda r_d)\right] \\
& -\frac{c_1}{6\lambda^2}\frac{\partial\lambda}{\partial\alpha_x}\left[\tfrac{1}{4}(c_3+c_5)^2\left(9\sin(\tfrac{1}{2}\lambda r_d)+\sin(\tfrac{3}{2}\lambda r_d)\right)+(c_4+c_6)^2\sin^3(\tfrac{1}{2}\lambda r_d)\right] \\
& +\frac{c_1}{6\lambda}\left[\tfrac{1}{4}2(c_3+c_5)\left(\frac{\partial c_3}{\partial\alpha_x}+\frac{\partial c_5}{\partial\alpha_x}\right)\left(9\sin(\tfrac{1}{2}\lambda r_d)+\sin(\tfrac{3}{2}\lambda r_d)\right)\right. \\
& +\tfrac{1}{4}(c_3+c_5)^2\left(9\frac{\partial\sin(\tfrac{1}{2}\lambda r_d)}{\partial\alpha_x}+\frac{\partial\sin(\tfrac{3}{2}\lambda r_d)}{\partial\alpha_x}\right)+2(c_4+c_6)\left(\frac{\partial c_4}{\partial\alpha_x}+\frac{\partial c_6}{\partial\alpha_x}\right)\sin^3(\tfrac{1}{2}\lambda r_d) \\
& \left.+3(c_4+c_6)^2\sin^2(\tfrac{1}{2}\lambda r_d)\frac{\partial\sin(\tfrac{1}{2}\lambda r_d)}{\partial\alpha_x}\right] \\
& +\frac{\partial\psi_P}{\partial\alpha_x}\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2+\psi_P\frac{\partial}{\partial\alpha_x}\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2 \\
= {} & \frac{1}{c_1}\frac{\partial c_1}{\partial\alpha_x}\left[\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2\phi(r)-\psi_P\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2\right] \\
& -\frac{1}{\lambda}\frac{\partial\lambda}{\partial\alpha_x}\left[\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2\phi(r)-\psi_P\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2\right] \\
& +\frac{c_1}{6\lambda}\left[\tfrac{1}{2}(c_3+c_5)\left(\frac{\partial c_3}{\partial\alpha_x}+\frac{\partial c_5}{\partial\alpha_x}\right)\left(9\sin(\tfrac{1}{2}\lambda r_d)+\sin(\tfrac{3}{2}\lambda r_d)\right)\right. \\
& +\tfrac{1}{4}(c_3+c_5)^2\left(\frac{9r_d}{2}\cos(\tfrac{1}{2}\lambda r_d)\frac{\partial\lambda}{\partial\alpha_x}+\frac{3r_d}{2}\cos(\tfrac{3}{2}\lambda r_d)\frac{\partial\lambda}{\partial\alpha_x}\right) \\
& +2(c_4+c_6)\left(\frac{\partial c_4}{\partial\alpha_x}+\frac{\partial c_6}{\partial\alpha_x}\right)\sin^3(\tfrac{1}{2}\lambda r_d) \\
& \left.+3(c_4+c_6)^2\sin^2(\tfrac{1}{2}\lambda r_d)\tfrac{1}{2}r_d\cos(\tfrac{1}{2}\lambda r_d)\frac{\partial\lambda}{\partial\alpha_x}\right] \\
& +\frac{\partial\psi_P}{\partial\alpha_x}\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2+\psi_P\frac{\partial}{\partial\alpha_x}\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&=\left(\frac{1}{c_1}\frac{\partial c_1}{\partial \alpha_x}-\frac{1}{\lambda}\frac{\partial \lambda}{\partial \alpha_x}\right)\left[\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2\phi(r)-\psi_P\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2\right]\\
&+\frac{c_1}{12\lambda}\left[(c_3+c_5)\left(\frac{\partial c_3}{\partial \alpha_x}+\frac{\partial c_5}{\partial \alpha_x}\right)\left(9\sin(\tfrac{1}{2}\lambda r_d)+\sin(\tfrac{3}{2}\lambda r_d)\right)\right.\\
&\left.+4(c_4+c_6)\left(\frac{\partial c_4}{\partial \alpha_x}+\frac{\partial c_6}{\partial \alpha_x}\right)\sin^3(\tfrac{1}{2}\lambda r_d)\right]\\
&+\frac{c_1 r_d}{16\lambda}\Big[(c_3+c_5)^2\left(3\cos(\tfrac{1}{2}\lambda r_d)+\cos(\tfrac{3}{2}\lambda r_d)\right)\\
&+4(c_4+c_6)^2\sin^2(\tfrac{1}{2}\lambda r_d)\cos(\tfrac{1}{2}\lambda r_d)\Big]\frac{\partial \lambda}{\partial \alpha_x}\\
&+\frac{\partial \psi_P}{\partial \alpha_x}\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2+\psi_P\frac{\partial}{\partial \alpha_x}\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2.
\end{aligned}
\tag{58}
$$

The derivative of R_1 of Eq. (31) is

$$
\begin{aligned}
\frac{\partial R_1}{\partial \alpha_x}&=\tfrac{1}{2}\Sigma_d\mu_+\frac{\partial \psi_+(\tfrac{1}{2}r_d)}{\partial \alpha_x}\\
&=\tfrac{1}{2}\Sigma_d\mu_+\left(\frac{\partial c_1}{\partial \alpha_x}\cos(\tfrac{1}{2}\lambda r_d)+c_1\frac{\partial\cos(\tfrac{1}{2}\lambda r_d)}{\partial \alpha_x}+\frac{\partial c_2}{\partial \alpha_x}\sin(\tfrac{1}{2}\lambda r_d)+c_2\frac{\partial\sin(\tfrac{1}{2}\lambda r_d)}{\partial \alpha_x}+\frac{\partial \psi_P}{\partial \alpha_x}\right)\\
&=\tfrac{1}{2}\Sigma_d\mu_+\left(\frac{\partial c_1}{\partial \alpha_x}\cos(\tfrac{1}{2}\lambda r_d)-\frac{c_1 r_d}{2}\sin(\tfrac{1}{2}\lambda r_d)\frac{\partial \lambda}{\partial \alpha_x}+\frac{\partial c_2}{\partial \alpha_x}\sin(\tfrac{1}{2}\lambda r_d)+\frac{c_2 r_d}{2}\cos(\tfrac{1}{2}\lambda r_d)\frac{\partial \lambda}{\partial \alpha_x}+\frac{\partial \psi_P}{\partial \alpha_x}\right)\\
&=\tfrac{1}{2}\Sigma_d\mu_+\left[\frac{\partial c_1}{\partial \alpha_x}\cos(\tfrac{1}{2}\lambda r_d)+\frac{\partial c_2}{\partial \alpha_x}\sin(\tfrac{1}{2}\lambda r_d)+\frac{\partial \psi_P}{\partial \alpha_x}+\frac{r_d}{2}\left(c_2\cos(\tfrac{1}{2}\lambda r_d)-c_1\sin(\tfrac{1}{2}\lambda r_d)\right)\frac{\partial \lambda}{\partial \alpha_x}\right].
\end{aligned}
\tag{59}
$$

Using Eq. (36), the derivative of ${}_2S$ of Eq. (33) is

$$
\begin{aligned}
\frac{\partial\,{}_2S}{\partial \alpha_x}&=\left(\frac{\partial N_i}{\partial \alpha_x}\overline{\nu(\nu-1)}_i\sigma_{f,i}+N_i\frac{\partial}{\partial \alpha_x}\overline{\nu(\nu-1)}_i\sigma_{f,i}+N_i\overline{\nu(\nu-1)}_i\frac{\partial \sigma_{f,i}}{\partial \alpha_x}\right)\chi^2\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2\phi(r)\\
&+\overline{\nu(\nu-1)}\Sigma_f\left(2\chi\frac{\partial \chi}{\partial \alpha_x}\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2\phi(r)+\chi^2\frac{\partial}{\partial \alpha_x}\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2\phi(r)\right).
\end{aligned}
\tag{60}
$$

Likewise, using Eq. (37), the derivative of ${}_2S_{s.f.}$ of Eq. (34) is

$$\begin{aligned}\frac{\partial\,{}_2S_{s.f.}}{\partial\alpha_x} &= \left[\frac{\partial N_i}{\partial\alpha_x} q_i \left(\frac{\overline{\nu(\nu-1)}}{\bar{\nu}}\right)_{s.f.,i} + \frac{N_i q_i}{\bar{\nu}_{s.f.,i}}\frac{\partial}{\partial\alpha_x}\overline{\nu(\nu-1)}_{s.f.,i}\right. \\ &\quad \left. -\frac{N_i q_i}{\bar{\nu}_{s.f.,i}}\left(\frac{\overline{\nu(\nu-1)}}{\bar{\nu}}\right)_{s.f.,i}\frac{\partial\bar{\nu}_{s.f.,i}}{\partial\alpha_x} + N_i\left(\frac{\overline{\nu(\nu-1)}}{\bar{\nu}}\right)_{s.f.,i}\frac{\partial q_i}{\partial\alpha_x}\right]\chi_{s.f.}^2\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2 \\ &\quad +\left(\frac{\overline{\nu(\nu-1)}}{\bar{\nu}}\right)_{s.f.} q\left(2\chi_{s.f.}\frac{\partial\chi_{s.f.}}{\partial\alpha_x}\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2 + \chi_{s.f.}^2\frac{\partial}{\partial\alpha_x}\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2\right).\end{aligned} \tag{61}$$

Using Eq. (35), the derivative of the induced-fission χ is

$$\frac{\partial\chi}{\partial\alpha_x} = \begin{cases}\dfrac{\nu\sigma_{f,i}N_i f_i}{\sum_{i'=1}^{I}\nu\sigma_{f,i'}N_{i'}f_{i'}}, & \alpha_x = \chi_i \\ 0, & \text{otherwise.}\end{cases} \tag{62}$$

This derivative is examined in detail in Appendix A. The derivative of the spontaneous-fission spectrum $\chi_{s.f.}$ is zero for all inputs; see Appendix B.

Finally, the derivative of the Feynman Y of Eq. (38) is

$$\begin{aligned}\frac{\partial Y}{\partial\alpha_x} &= \frac{1}{R_1}\left(\frac{\partial\,{}_2S}{\partial\alpha_x} + \frac{\partial\,{}_2S_{s.f.}}{\partial\alpha_x}\right) - \frac{{}_2S + {}_2S_{s.f.}}{R_1^2}\frac{\partial R_1}{\partial\alpha_x} \\ &= \frac{1}{R_1}\left(\frac{\partial\,{}_2S}{\partial\alpha_x} + \frac{\partial\,{}_2S_{s.f.}}{\partial\alpha_x} - Y\frac{\partial R_1}{\partial\alpha_x}\right).\end{aligned} \tag{63}$$

IV. Derivatives with Respect to the Slab Width

We seek the derivative of the Feynman Y of Eq. (38) with respect to the slab width r_d.

The derivative of ψ_P of Eq. (11) and the derivative of λ of Eq. (12) with respect to the slab width are zero.

We will need the derivatives of sines and cosines. They are simply

$$\frac{\partial\sin(a\lambda r_d)}{\partial r_d} = a\lambda\cos(a\lambda r_d) \tag{64}$$

and

$$\frac{\partial\cos(a\lambda r_d)}{\partial r_d} = -a\lambda\sin(a\lambda r_d), \tag{65}$$

where a is an arbitrary coefficient that does not depend on r_d.

The derivative of D of Eq. (15) is

$$\frac{\partial D}{\partial r_d} = -\mu_+\lambda \frac{\partial \cos(\lambda r_d)}{\partial r_d} - (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\frac{\partial \sin(\lambda r_d)}{\partial r_d}$$

$$= \lambda\left[\mu_+\lambda\sin(\lambda r_d) - (\Sigma_t - \tfrac{1}{2}\Sigma_s - \tfrac{1}{2}\chi\nu\Sigma_f)\cos(\lambda r_d)\right]. \tag{66}$$

The derivative of $1/D$ is

$$\frac{\partial}{\partial r_d}\left(\frac{1}{D}\right) = -\frac{1}{D^2}\frac{\partial D}{\partial r_d}. \tag{67}$$

The derivative of c_1 of Eq. (18) is

$$\frac{\partial c_1}{\partial r_d} = \psi_P \frac{\partial}{\partial r_d}\left(\frac{1}{D}\right)\left(\Sigma_t\sin(\tfrac{1}{2}\lambda r_d) + \mu_+\lambda\cos(\tfrac{1}{2}\lambda r_d)\right) + \frac{\psi_P}{D}\left(\Sigma_t\frac{\partial \sin(\tfrac{1}{2}\lambda r_d)}{\partial r_d} + \mu_+\lambda\frac{\partial \cos(\tfrac{1}{2}\lambda r_d)}{\partial r_d}\right)$$

$$= -\frac{\psi_P}{D^2}\frac{\partial D}{\partial r_d}\left(\Sigma_t\sin(\tfrac{1}{2}\lambda r_d) + \mu_+\lambda\cos(\tfrac{1}{2}\lambda r_d)\right) + \frac{\psi_P}{D}\left(\Sigma_t\tfrac{1}{2}\lambda\cos(\tfrac{1}{2}\lambda r_d) - \mu_+\lambda^2\tfrac{1}{2}\sin(\tfrac{1}{2}\lambda r_d)\right)$$

$$= -\frac{c_1}{D}\frac{\partial D}{\partial r_d} + \frac{\psi_P\lambda}{2D}\left(\Sigma_t\cos(\tfrac{1}{2}\lambda r_d) - \mu_+\lambda\sin(\tfrac{1}{2}\lambda r_d)\right). \tag{68}$$

The derivative of c_2 of Eq. (19) is

$$\frac{\partial c_2}{\partial r_d} = \psi_P \frac{\partial}{\partial r_d}\left(\frac{1}{D}\right)\left[\mu_+\lambda\sin(\tfrac{1}{2}\lambda r_d) - (\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d)\right]$$

$$+ \frac{\psi_P}{D}\left(\mu_+\lambda\frac{\partial \sin(\tfrac{1}{2}\lambda r_d)}{\partial r_d} - (\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)\frac{\partial \cos(\tfrac{1}{2}\lambda r_d)}{\partial r_d}\right)$$

$$= -\frac{\psi_P}{D^2}\frac{\partial D}{\partial r_d}\left[\mu_+\lambda\sin(\tfrac{1}{2}\lambda r_d) - (\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d)\right]$$

$$+ \frac{\psi_P}{D}\left(\mu_+\lambda^2\tfrac{1}{2}\cos(\tfrac{1}{2}\lambda r_d) + (\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)\tfrac{1}{2}\lambda\sin(\tfrac{1}{2}\lambda r_d)\right)$$

$$= -\frac{c_2}{D}\frac{\partial D}{\partial r_d} + \frac{\psi_P\lambda}{2D}\left(\mu_+\lambda\cos(\tfrac{1}{2}\lambda r_d) + (\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)\sin(\tfrac{1}{2}\lambda r_d)\right). \tag{69}$$

The derivative of c_3 of Eq. (24) is

$$\frac{\partial c_3}{\partial r_d} = \frac{-\Sigma_d}{2}\frac{\partial}{\partial r_d}\left(\frac{1}{D}\right)(\Sigma_s + \chi\nu\Sigma_f)\sin(\tfrac{1}{2}\lambda r_d) - \frac{\Sigma_d}{2D}(\Sigma_s + \chi\nu\Sigma_f)\frac{\partial \sin(\tfrac{1}{2}\lambda r_d)}{\partial r_d}$$

$$= \frac{\Sigma_d}{2D^2}\frac{\partial D}{\partial r_d}(\Sigma_s + \chi\nu\Sigma_f)\sin(\tfrac{1}{2}\lambda r_d) - \frac{\Sigma_d}{2D}(\Sigma_s + \chi\nu\Sigma_f)\tfrac{1}{2}\lambda\cos(\tfrac{1}{2}\lambda r_d)$$

$$= -\frac{c_3}{D}\frac{\partial D}{\partial r_d} - \frac{\Sigma_d\lambda}{4D}(\Sigma_s + \chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d). \tag{70}$$

The derivative of c_4 of Eq. (25) is

$$\begin{aligned}\frac{\partial c_4}{\partial r_d} &= \frac{-\Sigma_d}{2}\frac{\partial}{\partial r_d}\left(\frac{1}{D}\right)(\Sigma_s+\chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d) - \frac{\Sigma_d}{2D}(\Sigma_s+\chi\nu\Sigma_f)\frac{\partial\cos(\tfrac{1}{2}\lambda r_d)}{\partial r_d}\\ &= \frac{\Sigma_d}{2D^2}\frac{\partial D}{\partial r_d}(\Sigma_s+\chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d) + \frac{\Sigma_d}{2D}(\Sigma_s+\chi\nu\Sigma_f)\tfrac{1}{2}\lambda\sin(\tfrac{1}{2}\lambda r_d)\\ &= -\frac{c_4}{D}\frac{\partial D}{\partial r_d} + \frac{\Sigma_d\lambda}{4D}(\Sigma_s+\chi\nu\Sigma_f)\sin(\tfrac{1}{2}\lambda r_d).\end{aligned} \tag{71}$$

The derivative of c_5 of Eq. (29) is

$$\begin{aligned}\frac{\partial c_5}{\partial r_d} &= -\Sigma_d\frac{\partial}{\partial r_d}\left(\frac{1}{D}\right)\left[\mu_+\lambda\cos(\tfrac{1}{2}\lambda r_d) + (\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\sin(\tfrac{1}{2}\lambda r_d)\right]\\ &\quad -\frac{\Sigma_d}{D}\left[\mu_+\lambda\frac{\partial\cos(\tfrac{1}{2}\lambda r_d)}{\partial r_d} + (\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\frac{\partial\sin(\tfrac{1}{2}\lambda r_d)}{\partial r_d}\right]\\ &= \frac{\Sigma_d}{D^2}\frac{\partial D}{\partial r_d}\left[\mu_+\lambda\cos(\tfrac{1}{2}\lambda r_d) + (\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\sin(\tfrac{1}{2}\lambda r_d)\right]\\ &\quad -\frac{\Sigma_d}{D}\left[-\mu_+\lambda^2\tfrac{1}{2}\sin(\tfrac{1}{2}\lambda r_d) + (\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\tfrac{1}{2}\lambda\cos(\tfrac{1}{2}\lambda r_d)\right]\\ &= -\frac{c_5}{D}\frac{\partial D}{\partial r_d} + \frac{\Sigma_d\lambda}{2D}\left[\mu_+\lambda\sin(\tfrac{1}{2}\lambda r_d) - (\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d)\right].\end{aligned} \tag{72}$$

The derivative of c_6 of Eq. (30) is

$$\begin{aligned}\frac{\partial c_6}{\partial r_d} &= -\Sigma_d\frac{\partial}{\partial r_d}\left(\frac{1}{D}\right)\left[-\mu_+\lambda\sin(\tfrac{1}{2}\lambda r_d) + (\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d)\right]\\ &\quad -\frac{\Sigma_d}{D}\left[-\mu_+\lambda\frac{\partial\sin(\tfrac{1}{2}\lambda r_d)}{\partial r_d} + (\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\frac{\partial\cos(\tfrac{1}{2}\lambda r_d)}{\partial r_d}\right]\\ &= \frac{\Sigma_d}{D^2}\frac{\partial D}{\partial r_d}\left[-\mu_+\lambda\sin(\tfrac{1}{2}\lambda r_d) + (\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\cos(\tfrac{1}{2}\lambda r_d)\right]\\ &\quad -\frac{\Sigma_d}{D}\left[-\mu_+\lambda^2\tfrac{1}{2}\cos(\tfrac{1}{2}\lambda r_d) - (\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\tfrac{1}{2}\lambda\sin(\tfrac{1}{2}\lambda r_d)\right]\\ &= -\frac{c_6}{D}\frac{\partial D}{\partial r_d} + \frac{\Sigma_d\lambda}{2D}\left[\mu_+\lambda\cos(\tfrac{1}{2}\lambda r_d) + (\Sigma_t-\tfrac{1}{2}\Sigma_s-\tfrac{1}{2}\chi\nu\Sigma_f)\sin(\tfrac{1}{2}\lambda r_d)\right].\end{aligned} \tag{73}$$

The derivative of $\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2$ of Eq. (41) is

$$
\begin{aligned}
\frac{\partial}{\partial r_d}\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2 &= \frac{1}{8\lambda}\Bigg[2(c_3+c_5)\left(\frac{\partial c_3}{\partial r_d}+\frac{\partial c_5}{\partial r_d}\right)\left(\lambda r_d+\sin(\lambda r_d)\right)+(c_3+c_5)^2\left(\lambda+\frac{\partial \sin(\lambda r_d)}{\partial r_d}\right)\\
&\quad +2(c_4+c_6)\left(\frac{\partial c_4}{\partial r_d}+\frac{\partial c_6}{\partial r_d}\right)\left(\lambda r_d-\sin(\lambda r_d)\right)+(c_4+c_6)^2\left(\lambda-\frac{\partial \sin(\lambda r_d)}{\partial r_d}\right)\Bigg]\\
&= \frac{1}{8\lambda}\Bigg[2(c_3+c_5)\left(\frac{\partial c_3}{\partial r_d}+\frac{\partial c_5}{\partial r_d}\right)\left(\lambda r_d+\sin(\lambda r_d)\right)+\lambda(c_3+c_5)^2\left(1+\cos(\lambda r_d)\right)\\
&\quad +2(c_4+c_6)\left(\frac{\partial c_4}{\partial r_d}+\frac{\partial c_6}{\partial r_d}\right)\left(\lambda r_d-\sin(\lambda r_d)\right)+\lambda(c_4+c_6)^2\left(1-\cos(\lambda r_d)\right)\Bigg]\\
&= \frac{1}{4\lambda}\Bigg[(c_3+c_5)\left(\frac{\partial c_3}{\partial r_d}+\frac{\partial c_5}{\partial r_d}\right)\left(\lambda r_d+\sin(\lambda r_d)\right)\\
&\quad +(c_4+c_6)\left(\frac{\partial c_4}{\partial r_d}+\frac{\partial c_6}{\partial r_d}\right)\left(\lambda r_d-\sin(\lambda r_d)\right)\Bigg]\\
&\quad +\frac{1}{8}\left[(c_3+c_5)^2+(c_4+c_6)^2+\left((c_3+c_5)^2-(c_4+c_6)^2\right)\cos(\lambda r_d)\right].
\end{aligned}
\tag{74}
$$

The derivative of $\int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r)\right)^2 \phi(r)$ of Eq. (42) is

$$
\begin{aligned}
\frac{\partial}{\partial r_d}\int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r)\right)^2 \phi(r) &= \frac{\partial c_1}{\partial r_d}\frac{1}{6\lambda}\left[\tfrac{1}{4}(c_3+c_5)^2\left(9\sin(\tfrac{1}{2}\lambda r_d)+\sin(\tfrac{3}{2}\lambda r_d)\right)+(c_4+c_6)^2\sin^3(\tfrac{1}{2}\lambda r_d)\right] \\
&+\frac{c_1}{6\lambda}\left[\tfrac{1}{4}2(c_3+c_5)\left(\frac{\partial c_3}{\partial r_d}+\frac{\partial c_5}{\partial r_d}\right)\left(9\sin(\tfrac{1}{2}\lambda r_d)+\sin(\tfrac{3}{2}\lambda r_d)\right)\right. \\
&+\tfrac{1}{4}(c_3+c_5)^2\left(9\frac{\partial\sin(\tfrac{1}{2}\lambda r_d)}{\partial r_d}+\frac{\partial\sin(\tfrac{3}{2}\lambda r_d)}{\partial r_d}\right) \\
&+2(c_4+c_6)\left(\frac{\partial c_4}{\partial r_d}+\frac{\partial c_6}{\partial r_d}\right)\sin^3(\tfrac{1}{2}\lambda r_d) \\
&\left.+3(c_4+c_6)^2\sin^2(\tfrac{1}{2}\lambda r_d)\frac{\partial\sin(\tfrac{1}{2}\lambda r_d)}{\partial r_d}\right]+\psi_P\frac{\partial}{\partial r_d}\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2 \\
&=\frac{1}{c_1}\frac{\partial c_1}{\partial r_d}\left[\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2\phi(r)-\psi_P\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2\right] \\
&+\frac{c_1}{6\lambda}\left[\tfrac{1}{2}(c_3+c_5)\left(\frac{\partial c_3}{\partial r_d}+\frac{\partial c_5}{\partial r_d}\right)\left(9\sin(\tfrac{1}{2}\lambda r_d)+\sin(\tfrac{3}{2}\lambda r_d)\right)\right. \\
&+\tfrac{1}{4}(c_3+c_5)^2\left(9\tfrac{1}{2}\lambda\cos(\tfrac{1}{2}\lambda r_d)+\tfrac{3}{2}\lambda\cos(\tfrac{3}{2}\lambda r_d)\right) \\
&+2(c_4+c_6)\left(\frac{\partial c_4}{\partial r_d}+\frac{\partial c_6}{\partial r_d}\right)\sin^3(\tfrac{1}{2}\lambda r_d) \\
&\left.+3(c_4+c_6)^2\sin^2(\tfrac{1}{2}\lambda r_d)\tfrac{1}{2}\lambda\cos(\tfrac{1}{2}\lambda r_d)\right]+\psi_P\frac{\partial}{\partial r_d}\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2 \\
&=\frac{1}{c_1}\frac{\partial c_1}{\partial r_d}\left[\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2\phi(r)-\psi_P\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2\right] \\
&+\frac{c_1}{6\lambda}\left[\frac{1}{2}(c_3+c_5)\left(\frac{\partial c_3}{\partial r_d}+\frac{\partial c_5}{\partial r_d}\right)\left(9\sin(\tfrac{1}{2}\lambda r_d)+\sin(\tfrac{3}{2}\lambda r_d)\right)\right. \\
&+\frac{3\lambda}{8}(c_3+c_5)^2\left(3\cos(\tfrac{1}{2}\lambda r_d)+\cos(\tfrac{3}{2}\lambda r_d)\right) \\
&+2(c_4+c_6)\left(\frac{\partial c_4}{\partial r_d}+\frac{\partial c_6}{\partial r_d}\right)\sin^3(\tfrac{1}{2}\lambda r_d) \\
&\left.+\frac{3\lambda}{2}(c_4+c_6)^2\sin^2(\tfrac{1}{2}\lambda r_d)\cos(\tfrac{1}{2}\lambda r_d)\right]+\psi_P\frac{\partial}{\partial r_d}\int_{-r_d/2}^{r_d/2} dr\left(\phi^*(r)\right)^2 .
\end{aligned}
\tag{75}
$$

The derivative of R_1 of Eq. (31) is

$$\frac{\partial R_1}{\partial r_d} = \tfrac{1}{2}\Sigma_d \mu_+ \frac{\partial \psi_+(\tfrac{1}{2} r_d)}{\partial r_d}$$
$$= \tfrac{1}{2}\Sigma_d \mu_+ \left(\frac{\partial c_1}{\partial r_d}\cos(\tfrac{1}{2}\lambda r_d) + c_1 \frac{\partial \cos(\tfrac{1}{2}\lambda r_d)}{\partial r_d} + \frac{\partial c_2}{\partial r_d}\sin(\tfrac{1}{2}\lambda r_d) + c_2 \frac{\partial \sin(\tfrac{1}{2}\lambda r_d)}{\partial r_d} \right)$$
$$= \tfrac{1}{2}\Sigma_d \mu_+ \left(\frac{\partial c_1}{\partial r_d}\cos(\tfrac{1}{2}\lambda r_d) - c_1 \tfrac{1}{2}\lambda \sin(\tfrac{1}{2}\lambda r_d) + \frac{\partial c_2}{\partial r_d}\sin(\tfrac{1}{2}\lambda r_d) + c_2 \tfrac{1}{2}\lambda \cos(\tfrac{1}{2}\lambda r_d) \right)$$
$$= \tfrac{1}{2}\Sigma_d \mu_+ \left[\frac{\partial c_1}{\partial r_d}\cos(\tfrac{1}{2}\lambda r_d) + \frac{\partial c_2}{\partial r_d}\sin(\tfrac{1}{2}\lambda r_d) + \frac{\lambda}{2}\left(-c_1 \sin(\tfrac{1}{2}\lambda r_d) + c_2 \cos(\tfrac{1}{2}\lambda r_d)\right) \right]. \quad (76)$$

The derivative of ${}_2S$ of Eq. (33) is

$$\frac{\partial\, {}_2S}{\partial r_d} = \overline{\nu(\nu-1)}\Sigma_f \chi^2 \frac{\partial}{\partial r_d} \int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r)\right)^2 \phi(r) \quad (77)$$

and the derivative of ${}_2S_{s.f.}$ of Eq. (34) is

$$\frac{\partial\, {}_2S_{s.f.}}{\partial r_d} = \left(\frac{\overline{\nu(\nu-1)}}{\overline{\nu}} \right)_{s.f.} q\chi_{s.f.}^2 \frac{\partial}{\partial r_d} \int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r)\right)^2 . \quad (78)$$

The derivatives of the integrals are given by Eqs. (74) and (75).

Finally, the derivative of the Feynman *Y* of Eq. (38) is the same as Eq. (63),

$$\frac{\partial Y}{\partial r_d} = \frac{1}{R_1}\left(\frac{\partial\, {}_2S}{\partial r_d} + \frac{\partial\, {}_2S_{s.f.}}{\partial r_d} - Y \frac{\partial R_1}{\partial r_d} \right). \quad (79)$$

V. Test Problem

The test problem used a slab with width $r_d = 4$ cm. Regular S_2 ordinates, $\mu_\pm = \pm 1/\sqrt{3}$, were used. The material was plutonium with the composition given in Table I. Its mass density was 14 g/cm^3. The full SENSMG input file is listed in Appendix C.

Table I. Isotope Densities.

Isotope	Density (atoms/b·cm)
Pu-239	0.03385770516
Pu-240	0.001404851530

The neutron source rates from spontaneous fission of the two plutonium isotopes are given in Table II. These were computed using SOURCES4C (Ref. 16). Using the atom densities of Table I in Eq. (13), the total neutron source rate density for the material is q = 585.3096779 neutrons/cm^3·s. There are no (α,n) target isotopes.

Table II. Neutron Source Rates.

Isotope	Neutrons/s/(10^{24} atoms)
Pu-239	5.90346862E+00
Pu-240	4.16492268E+05

The collapsed one-group isotopic cross sections from MENDF71X are given in Table III. The one-group macroscopic cross sections for the material of Table I are given in Table IV.

Table III. Isotope Cross Sections.

Isotope	Cross section	Value (b)
Pu-239	σ_t	15.7163619565586
	$\nu\sigma_f$	10.2411266737305
	σ_s	10.1783768717624
	σ_a	5.5379850847962
	σ_f	3.49285663782976
	$\nu^{(a)}$	2.93202033052627
	σ_c	2.04811558813393
Pu-240	σ_t	13.8900165766247
	$\nu\sigma_f$	1.62998350059655
	σ_s	11.7932743116862
	σ_a	2.09674226493853
	σ_f	0.519536226613049
	$\nu^{(a)}$	3.13738179765193
	σ_c	1.58295129168442

(a) ν does not appear alone in the cross section data. It is $\nu\sigma_f$ divided by σ_f from the cross section data.

Table IV. Material Cross Sections.

Cross section	Value (cm^{-1})
Σ_t	0.551633360359619
$\nu\Sigma_f$	0.349030932243582
Σ_s	0.361184282597102
Σ_a	0.190449077762516

The first and second factorial moments of the multiplicity for induced thermal fission of Pu-239 and spontaneous fission of Pu-240 are given on Table V. Note that Pu-239 induced-fission $\overline{\nu}$ in Table V is different from the one-group ν in Table III, but these values are used independently, so the inconsistency causes no difficulty.

Table V. Fission Neutron Multiplicity Data[15]

Event	$\overline{\nu}$	$\overline{\nu(\nu-1)}$
Thermal fission of ^{239}Pu	2.8794	6.7728
Spontaneous fission of ^{240}Pu	2.1563	3.8242

The response function was $\Sigma_d = 0.009875877948$. It is shown in Table VI with other parameters that are presented in the text but do not appear in a previous table. It is an interesting quirk of the NDI that the spectrum weighting functions are isotope-dependent, but the difference in the values shown in Table VI has only a very small effect on the results.

Table VI. Other Parameters

Parameter	Value
μ_+	$1/\sqrt{3}$
q	585.3096779 neutrons/cm^3·s
Σ_d	0.009875877948
$\chi_{\mathrm{Pu}239}$	1.0
$\chi_{\mathrm{Pu}240}$	1.0
$f_{\mathrm{Pu}239}$	13.5759794880
$f_{\mathrm{Pu}240}$	13.5759619332
$\chi_{s.f.,\mathrm{Pu}239}$	1.0
$\chi_{s.f.\mathrm{Pu}240}$	1.0

The responses computed using the equations of Sec. II and the input values presented in this section are shown in Table VII and compared with values computed using SENSMG (with PARTISN used for the S_2 fluxes). A fine mesh spacing of 0.0005 cm was used, yielding 8000 meshes. A convergence criterion of 10^{-10} was used.

Table VII. Responses.

Response	Analytic	SENSMG	Difference
R_1	1.57256464E+02	1.572564E+02	–0.00001%
R_2	7.54409818E+02	7.544096E+02	–0.00003%
Y	4.79732153E+00	4.797320E+00	–0.00002%

Derivatives of material parameters needed to compute $\partial Y/\alpha_x$ appear in six combinations: $\partial\Sigma_t/\alpha_x$ appears in the derivative of c_1 [Eq. (51)]; $\partial(\Sigma_t-\Sigma_s-\chi\nu\Sigma_f)/\alpha_x$ appears in the derivatives of ψ_P, λ, and c_2 [Eqs. (44), (45), and (52)]; $\partial(\Sigma_t-\frac{1}{2}\Sigma_s-\frac{1}{2}\chi\nu\Sigma_f)/\alpha_x$ appears in the derivatives of D, c_5, and c_6 [Eqs. (49), (55), and (56)]; $\partial(\Sigma_s+\chi\nu\Sigma_f)/\alpha_x$ appears in the derivatives of c_3 and c_4 [Eqs. (53) and (54)]; $\partial q/\alpha_x$ appears only in the derivative of ψ_P [Eq. (44)]; and the derivative $\partial\chi/\alpha_x$ appears in the derivative of ${}_2S$ [Eq. (60)].

V.A.1. Sensitivity with Respect to the Pu-239 Total Cross Section

The macroscopic cross section for reaction *x* for a material is

$$\Sigma_x = \sum_{i=1}^{I} N_i \sigma_{x,i}. \tag{80}$$

The derivative of Σ_t with respect to the Pu-239 microscopic total cross section is

$$\frac{\partial \Sigma_t}{\partial \sigma_{t,\mathrm{Pu239}}} = N_{\mathrm{Pu239}}. \tag{81}$$

The derivatives of Σ_s and $\nu\Sigma_f$ are zero, so

$$\frac{\partial(\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)}{\partial \sigma_{t,\mathrm{Pu239}}} = \frac{\partial(\Sigma_t - \frac{1}{2}\Sigma_s - \frac{1}{2}\chi\nu\Sigma_f)}{\partial \sigma_{t,\mathrm{Pu239}}} = \frac{\partial \Sigma_t}{\partial \sigma_{t,\mathrm{Pu239}}} = N_{\mathrm{Pu239}}. \tag{82}$$

Also

$$\frac{\partial(\Sigma_s + \chi\nu\Sigma_f)}{\partial \sigma_{t,\mathrm{Pu239}}} = 0. \tag{83}$$

The derivative of the source rate density is zero,

$$\frac{\partial q}{\partial \sigma_{t,\mathrm{Pu239}}} = 0. \tag{84}$$

All of the derivatives used in the derivatives of ${}_2S$ and ${}_2S_{s.f.}$ [Eqs. (60) and (61)] are zero except for the last term:

$$\frac{\partial\, {}_2S}{\partial \sigma_{t,\mathrm{Pu239}}} = \overline{\nu(\nu-1)}\Sigma_f \chi^2 \frac{\partial}{\partial \sigma_{t,\mathrm{Pu239}}} \int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r)\right)^2 \phi(r), \tag{86}$$

$$\frac{\partial\, {}_2S_{s.f.}}{\partial \sigma_{t,\mathrm{Pu239}}} = \left(\frac{\overline{\nu(\nu-1)}}{\overline{\nu}}\right)_{s.f.} q\chi_{s.f.}^2 \frac{\partial}{\partial \sigma_{t,\mathrm{Pu239}}} \int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r)\right)^2. \tag{85}$$

Relative sensitivities from SENSMG are compared to analytic values in Table VIII. The relative sensitivity S_{R,α_x} of response *R* to parameter α_x is

$$S_{R,\alpha_x} \equiv \frac{\alpha_x}{R}\frac{\partial R}{\partial \alpha_x}. \tag{87}$$

The SENSMG relative sensitivities are extremely accurate for this problem.

Table VIII. Relative Sensitivities to $\sigma_{t,\mathrm{Pu239}}$.

Sensitivity	Analytic	SENSMG	Difference
$S_{R_1,\sigma_{t,\mathrm{Pu239}}}$	–3.832364E+01	–3.832363E+01	–0.00002%
$S_{R_2,\sigma_{t,\mathrm{Pu239}}}$	–1.138792E+02	–1.138792E+02	–0.00004%
$S_{Y,\sigma_{t,\mathrm{Pu239}}}$	–7.555561E+01	–7.555560E+01	–0.00001%

V.A.2. Sensitivity with Respect to the Pu-239 Fission Cross Section

The total macroscopic cross section for a material can be written

$$\Sigma_t = \Sigma_f + \sum \Sigma_{other} = \sum_{i=1}^{I} N_i \sigma_{f,i} + \sum \Sigma_{other}. \tag{88}$$

The derivative of Σ_t with respect to the Pu-239 microscopic fission cross section is

$$\frac{\partial \Sigma_t}{\partial \sigma_{f,\mathrm{Pu239}}} = N_{\mathrm{Pu239}}. \tag{89}$$

The macroscopic cross section $\nu\Sigma_f$ for a material obeys Eq. (80), so the derivative of $\nu\Sigma_f$ with respect to the Pu-239 fission cross section is

$$\frac{\partial \nu\Sigma_f}{\partial \sigma_{f,\mathrm{Pu239}}} = N_{\mathrm{Pu239}} \nu_{\mathrm{Pu239}}. \tag{90}$$

The derivative of Σ_s is zero, so

$$\frac{\partial (\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)}{\partial \sigma_{f,Pu239}} = N_{\mathrm{Pu239}} (1 - \chi \nu_{\mathrm{Pu239}}), \tag{91}$$

$$\frac{\partial (\Sigma_t - \frac{1}{2}\Sigma_s - \frac{1}{2}\chi\nu\Sigma_f)}{\partial \sigma_{f,Pu239}} = N_{\mathrm{Pu239}} (1 - \tfrac{1}{2} \chi \nu_{\mathrm{Pu239}}), \tag{92}$$

and

$$\frac{\partial (\Sigma_s + \chi\nu\Sigma_f)}{\partial \sigma_{f,\mathrm{Pu239}}} = \chi N_{\mathrm{Pu239}} \nu_{\mathrm{Pu239}}. \tag{93}$$

The derivative of the source rate density is zero,

$$\frac{\partial q}{\partial \sigma_{f,\mathrm{Pu239}}} = 0. \tag{94}$$

All of the derivatives used in the derivative of ${}_2S_{s.f.}$ [Eq. (61)] are zero except for the last term:

$$\frac{\partial\, {}_2S_{s.f.}}{\partial \sigma_{f,\mathrm{Pu239}}} = \left(\frac{\overline{\nu(\nu-1)}}{\overline{\nu}} \right)_{s.f.} q \chi_{s.f.}^2 \frac{\partial}{\partial \sigma_{f,\mathrm{Pu239}}} \int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r) \right)^2. \tag{95}$$

The derivative of the induced-fission moment ${}_2S$ of Eq. (60) is

$$\frac{\partial\, {}_2S}{\partial \sigma_{f,\mathrm{Pu239}}} = N_{\mathrm{Pu239}} \overline{\nu(\nu-1)}_{\mathrm{Pu239}} \chi^2 \int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r) \right)^2 \phi(r) + \overline{\nu(\nu-1)} \Sigma_f \chi^2 \frac{\partial}{\partial \sigma_{f,\mathrm{Pu239}}} \int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r) \right)^2 \phi(r). \tag{96}$$

Relative sensitivities from SENSMG are compared to analytic values in Table IX. The SENSMG relative sensitivities are extremely accurate for this problem.

Table IX. Relative Sensitivities to $\sigma_{f,\mathrm{Pu}239}$.

Sensitivity	Analytic	SENSMG	Difference
$S_{R_1,\sigma_{f,\mathrm{Pu}239}}$	1.993885E+01	1.993885E+01	0.00000%
$S_{R_2,\sigma_{f,\mathrm{Pu}239}}$	6.059542E+01	6.059541E+01	–0.00002%
$S_{Y,\sigma_{f,\mathrm{Pu}239}}$	4.065657E+01	4.065657E+01	0.00000%

V.A.3. Sensitivity with Respect to the Pu-239 ν

The derivative of the macroscopic material $\nu\Sigma_f$ with respect to the Pu-239 ν is

$$\frac{\partial \nu\Sigma_f}{\partial \nu_{\mathrm{Pu}239}} = N_{\mathrm{Pu}239}\sigma_{f,\mathrm{Pu}239}. \tag{97}$$

The derivatives of Σ_t and Σ_s are zero, so

$$\frac{\partial \Sigma_t}{\partial \nu_{\mathrm{Pu}239}} = 0, \tag{98}$$

$$\frac{\partial(\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)}{\partial \nu_{\mathrm{Pu}239}} = -\frac{\partial \chi\nu\Sigma_f}{\partial \nu_{\mathrm{Pu}239}} = -\chi N_{\mathrm{Pu}239}\sigma_{f,\mathrm{Pu}239}, \tag{99}$$

$$\frac{\partial(\Sigma_t - \frac{1}{2}\Sigma_s - \frac{1}{2}\chi\nu\Sigma_f)}{\partial \nu_{\mathrm{Pu}239}} = -\frac{1}{2}\frac{\partial \chi\nu\Sigma_f}{\partial \nu_{\mathrm{Pu}239}} = -\frac{1}{2}\chi N_{\mathrm{Pu}239}\sigma_{f,\mathrm{Pu}239}, \tag{100}$$

and

$$\frac{\partial(\Sigma_s + \chi\nu\Sigma_f)}{\partial \nu_{\mathrm{Pu}239}} = \frac{\partial \chi\nu\Sigma_f}{\partial \nu_{\mathrm{Pu}239}} = \chi N_{\mathrm{Pu}239}\sigma_{f,\mathrm{Pu}239}. \tag{101}$$

The derivative of the source rate density is zero,

$$\frac{\partial q}{\partial \nu_{\mathrm{Pu}239}} = 0. \tag{102}$$

All of the derivatives used in the derivative of ${}_2S_{s.f.}$ [Eq. (61)] are zero except for the last term:

$$\frac{\partial\, {}_2S_{s.f.}}{\partial \nu_{\mathrm{Pu}239}} = \left(\frac{\overline{\nu(\nu-1)}}{\overline{\nu}}\right)_{s.f.} q\chi_{s.f.}^2 \frac{\partial}{\partial \nu_{\mathrm{Pu}239}} \int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r)\right)^2. \tag{103}$$

We assume that $\overline{\nu(\nu-1)}_{\mathrm{Pu}239}$ is independent of $\nu_{\mathrm{Pu}239}$ and therefore

$$\frac{\partial \overline{\nu(\nu-1)}_{\mathrm{Pu}239}}{\partial \nu_{\mathrm{Pu}239}} = 0. \tag{104}$$

Then all of the derivatives used in the derivative of ${}_2S$ [Eq. (60)] are zero except for the last term:

$$\frac{\partial\, {}_2S}{\partial \nu_{\mathrm{Pu}239}} = \overline{\nu(\nu-1)}\Sigma_f\chi^2 \frac{\partial}{\partial \nu_{\mathrm{Pu}239}} \int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r)\right)^2 \phi(r). \tag{105}$$

Relative sensitivities from SENSMG are compared to analytic values in Table X. The SENSMG relative sensitivities are extremely accurate for this problem.

Table X. Relative Sensitivities to $\nu_{\mathrm{Pu}\,239}$.

Sensitivity	Analytic	SENSMG	Difference
$S_{R_1,\nu_{\mathrm{Pu}\,239}}$	2.845602E+01	2.845602E+01	–0.00001%
$S_{R_2,\nu_{\mathrm{Pu}\,239}}$	8.492993E+01	8.492992E+01	–0.00001%
$S_{Y,\nu_{\mathrm{Pu}\,239}}$	5.647390E+01	5.647390E+01	–0.00001%

V.A.4. Sensitivity with Respect to the Pu-239 Scattering Cross Section

The derivative of the material Σ_t and Σ_s with respect to the Pu-239 microscopic scattering cross section is

$$\frac{\partial \Sigma_t}{\partial \sigma_{s,\mathrm{Pu}\,239}} = \frac{\partial \Sigma_s}{\partial \sigma_{s,\mathrm{Pu}\,239}} = N_{\mathrm{Pu}\,239}. \tag{106}$$

The derivative of the macroscopic cross section $\nu\Sigma_f$ for a material is zero, so

$$\frac{\partial(\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)}{\partial \sigma_{s,\mathrm{Pu239}}} = 0, \tag{107}$$

$$\frac{\partial(\Sigma_t - \frac{1}{2}\Sigma_s - \frac{1}{2}\chi\nu\Sigma_f)}{\partial \sigma_{s,\mathrm{Pu239}}} = \tfrac{1}{2} N_{\mathrm{Pu}\,239}, \tag{108}$$

and

$$\frac{\partial(\Sigma_s + \chi\nu\Sigma_f)}{\partial \sigma_{s,\mathrm{Pu}\,239}} = N_{\mathrm{Pu}\,239}. \tag{109}$$

The derivative of the source rate density is zero,

$$\frac{\partial q}{\partial \sigma_{s,\mathrm{Pu}\,239}} = 0. \tag{110}$$

All of the derivatives used in the derivatives of ${}_2S$ and ${}_2S_{s.f.}$ [Eqs. (60) and (61)] are zero except for the last term:

$$\frac{\partial\, {}_2S}{\partial \sigma_{s,\mathrm{Pu}\,239}} = \overline{\nu(\nu-1)}\Sigma_f \chi^2 \frac{\partial}{\partial \sigma_{s,\mathrm{Pu}\,239}} \int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r)\right)^2 \phi(r), \tag{111}$$

$$\frac{\partial\, {}_2S_{s.f.}}{\partial \sigma_{s,\mathrm{Pu}\,239}} = \left(\frac{\overline{\nu(\nu-1)}}{\overline{\nu}}\right)_{s.f.} q\chi_{s.f.}^2 \frac{\partial}{\partial \sigma_{s,\mathrm{Pu}\,239}} \int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r)\right)^2. \tag{112}$$

Relative sensitivities from SENSMG are compared to analytic values in Table XI. The SENSMG relative sensitivities are extremely accurate for this problem.

Table XI. Relative Sensitivities to $\sigma_{s,\mathrm{Pu}\,239}$.

Sensitivity	Analytic	SENSMG	Difference
$S_{R_1,\sigma_{s,\mathrm{Pu}\,239}}$	3.462156E+00	3.462155E+00	–0.00002%
$S_{R_2,\sigma_{s,\mathrm{Pu}\,239}}$	1.065800E+01	1.065800E+01	–0.00004%
$S_{Y,\sigma_{s,\mathrm{Pu}\,239}}$	7.195848E+00	7.195847E+00	–0.00002%

V.A.5. Sensitivity with Respect to the Pu-240 Induced-Fission Spectrum

For this one-group problem, PARTISN constructs the material induced-fission spectrum χ using Eq. (36):

$$\chi = \frac{\chi_{\mathrm{Pu239}} \nu\sigma_{f,\mathrm{Pu239}} N_{\mathrm{Pu239}} f_{\mathrm{Pu239}} + \chi_{\mathrm{Pu240}} \nu\sigma_{f,\mathrm{Pu240}} N_{\mathrm{Pu240}} f_{\mathrm{Pu240}}}{\nu\sigma_{f,\mathrm{Pu239}} N_{\mathrm{Pu239}} f_{\mathrm{Pu239}} + \nu\sigma_{f,\mathrm{Pu240}} N_{\mathrm{Pu240}} f_{\mathrm{Pu240}}}. \tag{113}$$

The unnormalized derivative of χ with respect to the Pu-240 fission spectrum is Eq. (62):

$$\frac{\partial \chi}{\partial \chi_{\mathrm{Pu240}}} = \frac{\nu\sigma_{f,\mathrm{Pu240}} N_{\mathrm{Pu240}} f_{\mathrm{Pu240}}}{\nu\sigma_{f,\mathrm{Pu239}} N_{\mathrm{Pu239}} f_{\mathrm{Pu239}} + \nu\sigma_{f,\mathrm{Pu240}} N_{\mathrm{Pu240}} f_{\mathrm{Pu240}}}. \tag{114}$$

The derivatives of all of the material cross sections with respect to the Pu-240 fission spectrum are zero, so

$$\frac{\partial(\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)}{\partial \chi_{\mathrm{Pu240}}} = -\frac{\partial(\chi\nu\Sigma_f)}{\partial \chi_{\mathrm{Pu240}}} = -\frac{\nu\sigma_{f,\mathrm{Pu240}} N_{\mathrm{Pu240}} f_{\mathrm{Pu240}} \nu\Sigma_f}{\nu\sigma_{f,\mathrm{Pu239}} N_{\mathrm{Pu239}} f_{\mathrm{Pu239}} + \nu\sigma_{f,\mathrm{Pu240}} N_{\mathrm{Pu240}} f_{\mathrm{Pu240}}}, \tag{115}$$

$$\frac{\partial(\Sigma_t - \frac{1}{2}\Sigma_s - \frac{1}{2}\chi\nu\Sigma_f)}{\partial \chi_{\mathrm{Pu240}}} = -\frac{\partial(\frac{1}{2}\chi\nu\Sigma_f)}{\partial \chi_{\mathrm{Pu240}}} = -\frac{1}{2}\frac{\nu\sigma_{f,\mathrm{Pu240}} N_{\mathrm{Pu240}} f_{\mathrm{Pu240}} \nu\Sigma_f}{\nu\sigma_{f,\mathrm{Pu239}} N_{\mathrm{Pu239}} f_{\mathrm{Pu239}} + \nu\sigma_{f,\mathrm{Pu240}} N_{\mathrm{Pu240}} f_{\mathrm{Pu240}}}, \tag{116}$$

and

$$\frac{\partial(\Sigma_s + \chi\nu\Sigma_f)}{\partial \chi_{\mathrm{Pu240}}} = \frac{\partial(\chi\nu\Sigma_f)}{\partial \chi_{\mathrm{Pu240}}} = \frac{\nu\sigma_{f,\mathrm{Pu240}} N_{\mathrm{Pu240}} f_{\mathrm{Pu240}} \nu\Sigma_f}{\nu\sigma_{f,\mathrm{Pu239}} N_{\mathrm{Pu239}} f_{\mathrm{Pu239}} + \nu\sigma_{f,\mathrm{Pu240}} N_{\mathrm{Pu240}} f_{\mathrm{Pu240}}}. \tag{117}$$

The derivative of the source rate density is zero,

$$\frac{\partial q}{\partial \chi_{\mathrm{Pu240}}} = 0. \tag{118}$$

All of the derivatives used in the derivative of ${}_2S_{s.f.}$ [Eq. (61)] are zero except for the last term:

$$\frac{\partial\, {}_2S_{s.f.}}{\partial N_{\mathrm{Pu239}}} = \left(\frac{\overline{\nu(\nu-1)}}{\bar{\nu}}\right)_{s.f.} q\chi_{s.f.}^2 \frac{\partial}{\partial \chi_{\mathrm{Pu240}}} \int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r)\right)^2. \tag{119}$$

The derivative of the induced-fission moment ${}_2S$ of Eq. (60) is

$$\frac{\partial\, {}_2S}{\partial \chi_{\mathrm{Pu240}}} = \overline{\nu(\nu-1)}\Sigma_f \left(2\chi \frac{\partial \chi}{\partial \chi_{\mathrm{Pu240}}} \int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r)\right)^2 \phi(r) + \chi^2 \frac{\partial}{\partial \chi_{\mathrm{Pu240}}} \int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r)\right)^2 \phi(r)\right). \tag{120}$$

Relative sensitivities from SENSMG are compared to analytic values in Table XII. The SENSMG relative sensitivities are extremely accurate for this problem.

Table XII. Relative Sensitivities to χ_{Pu240} (Unnormalized).

Sensitivity	Analytic	SENSMG	Difference
$S_{R_1,\chi_{\mathrm{Pu240}}}$	1.879239E-01	1.879239E-01	0.00000%
$S_{R_2,\chi_{\mathrm{Pu240}}}$	5.736636E-01	5.736636E-01	–0.00001%
$S_{Y,\chi_{\mathrm{Pu240}}}$	3.857397E-01	3.857397E-01	–0.00001%

V.A.6. Sensitivity with Respect to the Pu-239 Atom Density

The derivative of Σ_t with respect to the Pu-239 microscopic atom density is

$$\frac{\partial \Sigma_t}{\partial N_{\mathrm{Pu239}}} = \sigma_{t,\mathrm{Pu239}}. \tag{121}$$

The derivatives of Σ_s and $\nu\Sigma_f$ are likewise $\sigma_{s,\mathrm{Pu239}}$ and $\nu\sigma_{f,\mathrm{Pu239}}$, so

$$\frac{\partial(\Sigma_t - \Sigma_s - \chi\nu\Sigma_f)}{\partial N_{\mathrm{Pu239}}} = \sigma_{t,\mathrm{Pu239}} - \sigma_{s,\mathrm{Pu239}} - \chi\nu\sigma_{f,\mathrm{Pu239}}, \tag{122}$$

$$\frac{\partial(\Sigma_t - \frac{1}{2}\Sigma_s - \frac{1}{2}\chi\nu\Sigma_f)}{\partial N_{\mathrm{Pu239}}} = \sigma_{t,\mathrm{Pu239}} - \tfrac{1}{2}\sigma_{s,\mathrm{Pu239}} - \tfrac{1}{2}\chi\nu\sigma_{f,\mathrm{Pu239}}, \tag{123}$$

$$\frac{\partial(\Sigma_s + \chi\nu\Sigma_f)}{\partial N_{\mathrm{Pu239}}} = \sigma_{s,\mathrm{Pu239}} + \chi\nu\sigma_{f,\mathrm{Pu239}}. \tag{124}$$

The derivative of the source rate density is

$$\frac{\partial q}{\partial N_{\mathrm{Pu239}}} = q_{\mathrm{Pu239}}, \tag{125}$$

the neutron source rate per 10^{24} atoms of Pu-239. All of the derivatives used in the derivative of ${}_2S_{s.f.}$ [Eq. (61)] are zero except for the last term:

$$\frac{\partial\, {}_2S_{s.f.}}{\partial N_{\mathrm{Pu239}}} = \left(\frac{\overline{\nu(\nu-1)}}{\bar{\nu}}\right)_{s.f.} q\chi_{s.f.}^2 \frac{\partial}{\partial N_{\mathrm{Pu239}}} \int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r)\right)^2. \tag{126}$$

The derivative of the induced-fission moment ${}_2S$ of Eq. (60) is

$$\frac{\partial\, {}_2S}{\partial N_{\mathrm{Pu239}}} = \overline{\nu(\nu-1)}_{\mathrm{Pu239}}\sigma_{f,\mathrm{Pu239}}\chi^2 \int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r)\right)^2 \phi(r) + \overline{\nu(\nu-1)}\Sigma_f\chi^2 \frac{\partial}{\partial N_{\mathrm{Pu239}}} \int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r)\right)^2 \phi(r). \tag{127}$$

Relative sensitivities from SENSMG are compared to analytic values in Table XIII. The SENSMG relative sensitivities are extremely accurate for this problem.

Table XIII. Relative Sensitivities to N_{Pu239}.

Sensitivity	Analytic	SENSMG	Difference
$S_{R_1,N_{\mathrm{Pu239}}}$	1.841439E+01	1.841439E+01	–0.00003%
$S_{R_2,N_{\mathrm{Pu239}}}$	5.643495E+01	5.643495E+01	–0.00001%
$S_{Y,N_{\mathrm{Pu239}}}$	3.802056E+01	3.802056E+01	0.00000%

V.A.7. Sensitivity with Respect to the Pu-240 Atom Density

The equations of Sec. V.A.6 apply with Pu239 changed to Pu240 except Eqs. (126) and (127). All of the derivatives used in the derivative of ${}_2S$ [Eq. (60)] are zero except for the last term:

$$\frac{\partial\, {}_2S}{\partial N_{\mathrm{Pu240}}} = \overline{\nu(\nu-1)}\Sigma_f\chi^2 \frac{\partial}{\partial N_{\mathrm{Pu240}}} \int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r)\right)^2 \phi(r). \tag{128}$$

The derivative of the induced-fission moment ${}_2S_{s.f.}$ of Eq. (61) is

$$\frac{\partial\, {}_2S_{s.f.}}{\partial N_{\mathrm{Pu}240}} = \left(\frac{\overline{\nu(\nu-1)}}{\overline{\nu}}\right)_{s.f.,\mathrm{Pu}240} q_{\mathrm{Pu}240}\chi^2_{s.f.} \int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r)\right)^2 + \left(\frac{\overline{\nu(\nu-1)}}{\overline{\nu}}\right)_{s.f.} q\chi^2_{s.f.} \frac{\partial_2}{\partial N_{\mathrm{Pu}240}} \int_{-r_d/2}^{r_d/2} dr \left(\phi^*(r)\right)^2 . \quad (129)$$

Relative sensitivities from SENSMG are compared to analytic values in Table XIV. The SENSMG relative sensitivities are extremely accurate for this problem.

Table XIV. Relative Sensitivities to $N_{\mathrm{Pu}\,240}$.

Sensitivity	Analytic	SENSMG	Difference
$S_{R_1,N_{\mathrm{Pu}\,240}}$	1.141884E+00	1.141884E+00	–0.00004%
$S_{R_2,N_{\mathrm{Pu}\,240}}$	1.442549E+00	1.442549E+00	–0.00001%
$S_{Y,N_{\mathrm{Pu}\,240}}$	3.006647E-01	3.006647E-01	0.00000%

V.A.8. Sensitivity with Respect to the Material Mass Density

The relative sensitivity of a response to the material mass density ρ is the sum of the relative sensitivities of the response to the isotopic atom densities.[13] For this problem, that means

$$S_{R,\rho} = S_{R,N_{\mathrm{Pu}\,239}} + S_{R,N_{\mathrm{Pu}\,240}} . \quad (130)$$

Equation (130) was used for the analytic sensitivities. The adjoint-based equation of Refs. 8 and 9 was used for the SENSMG sensitivities.

Relative sensitivities from SENSMG are compared to analytic values in Table XV. The SENSMG relative sensitivities are extremely accurate for this problem.

Table XV. Relative Sensitivities to ρ.

Sensitivity	Analytic	SENSMG	Difference
$S_{R_1,\rho}$	1.955628E+01	1.955628E+01	0.00000%
$S_{R_2,\rho}$	5.787750E+01	5.787750E+01	–0.00001%
$S_{Y,\rho}$	3.832122E+01	3.832122E+01	–0.00001%

V.A.9. Derivative with Respect to the Slab Width

Derivatives from SENSMG are compared to analytic values in Table XVI. The SENSMG derivatives are extremely accurate for this problem.

Table XVI. Derivatives with Respect to r_d.

Sensitivity	Analytic	SENSMG	Difference
$\partial R_1/\partial r_d$	7.688378E+02	7.688377E+02	–0.00002%
$\partial R_2/\partial r_d$	1.091584E+04	1.091583E+04	–0.00008%
$\partial Y/\partial r_d$	4.595981E+01	4.595979E+01	–0.00004%

An equation for the adjoint-based derivative of the Feynman Y to interface locations and the outer boundary has yet to be derived formally. Presently, SENSMG uses a straightforward extension of the equation for the derivative of the leakage,[17,18] applying a similar formula to the adjoint-based equation of Refs. 8 and 9. This is a work in progress.

VI. Summary and Future Work

This report has presented analytic derivatives of the Feynman Y for a homogeneous slab with isotropic scattering and S_2 quadrature, and it has compared sensitivities from SENSMG with the analytic sensitivities for several isotope cross sections, Pu-239 ν, Pu-240 induced fission χ, isotope atom densities, the material mass density, and the slab width. The SENSMG results are all extremely accurate. Sensitivities for any other material parameter can also be compared, except that SENSMG does not presently compute sensitivities with respect to the second factorial moment of the induced-fission multiplicity distribution, the first or second factorial moment of the spontaneous-fission multiplicity distribution, or the spontaneous-fission source rate per atom of a source isotope.

The derivative with respect to interface locations is a particular capability we are addressing.

It is possible to derive a multigroup, multiregion analytic solution for S_2 transport in a slab.[19] However, it is not clear to this author how to derive analytic derivatives from the solution given in Ref. 19. This would make a good project for a student.

This work represents just one verification of SENSMG's new multiplicity sensitivity capability. More thorough verification tests will be realistic and multigroup. This report only *verifies* (in part) the SENSMG calculation of the Feynman Y and its sensitivities. It does not *validate* the SENSMG calculation of the Feynman Y.

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APPENDIX A
DERIVATIVES OF THE INDUCED-FISSION SPECTRUM χ

For a G-group problem using the NDI and a χ vector, PARTISN constructs the material fission spectrum χ using

$$\chi^g = \frac{\sum_{i=1}^{I} \chi_i^g \sum_{g'=1}^{G} \nu\sigma_{f,i}^{g'} N_i f_i^{g'}}{\sum_{i=1}^{I}\sum_{g'=1}^{G} \nu\sigma_{f,i}^{g'} N_i f_i^{g'}}. \tag{A.1}$$

where f_i^g is the group-dependent spectrum weighting function and I is the number of fissionable isotopes. For a one-group problem, Eq. (A.1) becomes Eq. (35), repeated here:

$$\chi = \frac{\sum_{i=1}^{I} \chi_i \nu\sigma_{f,i} N_i f_i}{\sum_{i=1}^{I} \nu\sigma_{f,i} N_i f_i}. \tag{A.2}$$

The derivative of χ with respect to the fission spectrum of isotope i is Eq. (62),

$$\frac{\partial\chi}{\partial\chi_i} = \frac{\nu\sigma_{f,i} N_i f_i}{\sum_{i'=1}^{I} \nu\sigma_{f,i'} N_{i'} f_{i'}}. \tag{A.3}$$

The derivative of χ with respect to the ν of isotope i is

$$\begin{aligned}\frac{\partial\chi}{\partial\nu_i} &= \frac{\chi_i \sigma_{f,i} N_i f_i}{\sum_{i'=1}^{I} \nu\sigma_{f,i'} N_{i'} f_{i'}} - \frac{\sum_{i'=1}^{I} \chi_{i'} \nu\sigma_{f,i'} N_{i'} f_{i'}}{\sum_{i'=1}^{I} \nu\sigma_{f,i'} N_{i'} f_{i'}} \frac{\sigma_{f,i} N_i f_i}{\sum_{i'=1}^{I} \nu\sigma_{f,i'} N_{i'} f_{i'}} \\ &= \left(\chi_i - \chi\right) \frac{\sigma_{f,i} N_i f_i}{\sum_{i'=1}^{I} \nu\sigma_{f,i'} N_{i'} f_{i'}}.\end{aligned} \tag{A.4}$$

The derivative of χ with respect to the fission cross section of isotope i is

$$\begin{aligned}\frac{\partial\chi}{\partial\sigma_{f,i}} &= \frac{\chi_i \nu_i N_i f_i}{\sum_{i'=1}^{I} \nu\sigma_{f,i'} N_{i'} f_{i'}} - \frac{\sum_{i'=1}^{I} \chi_{i'} \nu\sigma_{f,i'} N_{i'} f_{i'}}{\sum_{i'=1}^{I} \nu\sigma_{f,i'} N_{i'} f_{i'}} \frac{\nu_i N_i f_i}{\sum_{i'=1}^{I} \nu\sigma_{f,i'} N_{i'} f_{i'}} \\ &= \left(\chi_i - \chi\right) \frac{\nu_i N_i f_i}{\sum_{i'=1}^{I} \nu\sigma_{f,i'} N_{i'} f_{i'}}.\end{aligned} \tag{A.5}$$

The derivative of χ with respect to the atom density of isotope i is

$$\frac{\partial \chi}{\partial N_i} = \frac{\chi_i \nu\sigma_{f,i} f_i}{\sum_{i'=1}^{I} \nu\sigma_{f,i'} N_{i'} f_{i'}} - \frac{\sum_{i'=1}^{I} \chi_{i'} \nu\sigma_{f,i'} N_{i'} f_{i'}}{\sum_{i'=1}^{I} \nu\sigma_{f,i'} N_{i'} f_{i'}} \frac{\nu\sigma_{f,i} f_i}{\sum_{i'=1}^{I} \nu\sigma_{f,i'} N_{i'} f_{i'}}$$

$$= \left(\chi_i - \chi\right) \frac{\nu\sigma_{f,i} f_i}{\sum_{i'=1}^{I} \nu\sigma_{f,i'} N_{i'} f_{i'}}. \tag{A.6}$$

In the one-group problem, $\chi = \chi_i = 1$ in the unperturbed case, so $\partial\chi/\partial\nu_i$, $\partial\chi/\partial\sigma_{f,i}$, and $\partial\chi/\partial N_i$ are all zero.

In the multigroup case, however, these derivatives are not zero. To take one example, the derivative of χ^g of Eq. (A.1) with respect to the atom density of isotope i is

$$\frac{\partial \chi^g}{\partial N_i} = \frac{\chi_i^g \sum_{g'=1}^{G} \nu\sigma_{f,i}^{g'} f_i^{g'}}{\sum_{i'=1}^{I}\sum_{g'=1}^{G} \nu\sigma_{f,i'}^{g'} N_{i'} f_{i'}^{g'}} - \frac{\sum_{i'=1}^{I} \chi_{i'}^g \sum_{g'=1}^{G} \nu\sigma_{f,i'}^{g'} N_{i'} f_{i'}^{g'}}{\sum_{i'=1}^{I}\sum_{g'=1}^{G} \nu\sigma_{f,i'}^{g'} N_{i'} f_{i'}^{g'}} \frac{\sum_{g'=1}^{G} \nu\sigma_{f,i}^{g'} f_i^{g'}}{\sum_{i'=1}^{I}\sum_{g'=1}^{G} \nu\sigma_{f,i'}^{g'} N_{i'} f_{i'}^{g'}}$$

$$= \left(\chi_i^g - \chi^g\right) \frac{\sum_{g'=1}^{G} \nu\sigma_{f,i}^{g'} f_i^{g'}}{\sum_{i'=1}^{I}\sum_{g'=1}^{G} \nu\sigma_{f,i'}^{g'} N_{i'} f_{i'}^{g'}}, \tag{A.7}$$

which is not zero for $I > 1$. Equation (A.7) gives the unconstrained derivative.

Using $N_i = a_i N$, where a_i is the atom fraction of isotope i and N is the atom density of the material, in Eq. (A.1) yields

$$\chi^g = \frac{\sum_{i=1}^{I} \chi_i^g \sum_{g'=1}^{G} \nu\sigma_{f,i}^{g'} a_i N f_i^{g'}}{\sum_{i=1}^{I}\sum_{g'=1}^{G} \nu\sigma_{f,i}^{g'} a_i N f_i^{g'}} = \frac{\sum_{i=1}^{I} \chi_i^g \sum_{g'=1}^{G} \nu\sigma_{f,i}^{g'} a_i f_i^{g'}}{\sum_{i=1}^{I}\sum_{g'=1}^{G} \nu\sigma_{f,i}^{g'} a_i f_i^{g'}}. \tag{A.8}$$

Therefore the derivative of χ^g with respect to the material atom density is

$$\frac{\partial \chi^g}{\partial N} = 0, \tag{A.9}$$

and this conclusion holds in the one-group case.

APPENDIX B
DERIVATIVES OF THE SPONTANEOUS-FISSION SPECTRUM $\chi_{s.f.}$

For a G-group problem, SENSMG constructs the material spontaneous-fission spectrum $\chi_{s.f.}^{g}$ using

$$\chi_{s.f.}^{g} = \frac{\sum_{i=1}^{I} q_i^g N_i}{\sum_{i=1}^{I} \sum_{g'=1}^{G} q_i^{g'} N_i}, \tag{B.1}$$

For a one-group problem, Eq. (B.1) becomes

$$\chi_{s.f.} = \frac{\sum_{i=1}^{I} q_i N_i}{\sum_{i=1}^{I} q_i N_i} = 1. \tag{B.2}$$

Clearly, for the one-group problem, the derivative of $\chi_{s.f.}$ with respect to all input parameters is zero.

In the multigroup case, the derivative of $\chi_{s.f.}^{g}$ with respect to the atom density of isotope i is

$$\begin{aligned}\frac{\partial \chi_{s.f.}^{g}}{\partial N_i} &= \frac{q_i^g}{\sum_{i'=1}^{I} \sum_{g'=1}^{G} q_{i'}^{g'} N_{i'}} - \frac{\sum_{i'=1}^{I} q_{i'}^{g} N_{i'}}{\sum_{i'=1}^{I} \sum_{g'=1}^{G} q_{i'}^{g'} N_{i'}} \frac{\sum_{g'=1}^{G} q_i^{g'}}{\sum_{i'=1}^{I} \sum_{g'=1}^{G} q_{i'}^{g'} N_{i'}} \\ &= \left(\frac{q_i^g}{\sum_{g'=1}^{G} q_i^{g'}} - \chi_{s.f.}^{g} \right) \frac{\sum_{g'=1}^{G} q_i^{g'}}{\sum_{i'=1}^{I} \sum_{g'=1}^{G} q_{i'}^{g'} N_{i'}},\end{aligned} \tag{B.3}$$

which is not zero for $I > 1$. Likewise, the derivative of $\chi_{s.f.}^{g}$ with respect to the source rate density in group g' due to isotope i is

$$\begin{aligned}\frac{\partial \chi_{s.f.}^{g}}{\partial q_i^{g'}} &= \delta_{gg'} \frac{N_i}{\sum_{i'=1}^{I} \sum_{g''=1}^{G} q_{i'}^{g''} N_{i'}} - \frac{\sum_{i'=1}^{I} q_{i'}^{g} N_{i'}}{\sum_{i'=1}^{I} \sum_{g''=1}^{G} q_{i'}^{g''} N_{i'}} \frac{N_i}{\sum_{i'=1}^{I} \sum_{g''=1}^{G} q_{i'}^{g''} N_{i'}} \\ &= \left(\delta_{gg'} - \chi_{s.f.}^{g} \right) \frac{N_i}{\sum_{i'=1}^{I} \sum_{g''=1}^{G} q_{i'}^{g''} N_{i'}}.\end{aligned} \tag{B.4}$$

Equations (B.3) and (B.4) give the constrained derivatives.

APPENDIX C
SENSMG INPUT FILE FOR THE TEST PROBLEM

The SENSMG source code was modified to run this problem using 0.0005 cm/mesh.

```
two-isotope slab
slab feyny
mendf71x
1 / no of materials
1 94239 -0.96 94240 -0.04  /
-14.00 / densities
1 / no of shells
 0.  4.  /
 1          / material nos
 0          / number of edit points
 0          / number of reaction-rate ratios
```

The following command line was used to run the input file above:

```
${SENSMG} -i slab -fissdata 2 -srcacc_no for+adj -misc no -epsi 1.e-10
-isn 2 -isct 0 -ngroup 1 -np 1 -chinorm none
```